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# ELECTROMAGNETIC PROPERTIES OF NUCLEAR SYSTEMS ON MESON THEORY

BY

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## INTRODUCTION

The present paper is intended as a sequel to the authors' previous work "On the field theory of nuclear forces" [1]. In the latter, the question was discussed, whether it would be possible in a well-defined way to delimit a domain in which the meson field theory, in spite of the fundamental difficulties of its quantization, would still yield consistent results. For this purpose, a point of view analogous to the "correspondence" treatment of quantum electrodynamics was adopted and a general prescription could be formulated according to which a consistent interpretation of the formalism may be obtained. For the application of this prescription to the problem of nuclear forces, a first requirement is to separate the "static" part of these forces, for the same reason as the Coulomb forces in electrodynamics must be separated from the radiation field before a correspondence treatment of the latter can be arrived at. It was then shown that such a separation of the static forces could simply be effected by a canonical transformation, and the application of the general criterium mentioned above to the expression obtained in this way for the static forces led to the adoption of the so-called "mixed" meson theory, *i. e.* a mixture of vector and pseudoscalar meson fields with properly chosen intensities of the corresponding nuclear sources [2]. While the consequences of the point of view just sketched for the theory of $\beta$-disintegration and meson life-time have been fully discussed [3], its bearing on the electromagnetic properties of nuclear systems remained to be examined: this is the program of the present paper.

The completion of this work, however, has been much delayed, partly due to fortuitous circumstances, but partly also on account of the development of the subject itself. In fact, it was pointed out in the meantime by one of us [4] that the mixed theory appeared as an elementary (*i. e.* not mixed) form of meson theory in a five-dimensional representation. A subsequent survey of the formal possibilities for five-dimensional meson field theories even showed that the mixed theory was the only combination of four-dimensional types of meson fields possessing the property just mentioned of coinciding with an irreducible five-dimensional type of field [5]. Two possible interpretations of such a five-dimensional representation were discussed; of these, it appeared that the projective interpretation was especially well-suited for the incorporation of the interaction with the electromagnetic field [6]. In this connexion, the main formulae concerning this interaction have already been published [6] and applied to the problem of the photo-effect of the deuteron [7]. On the other hand, this and other problems involving electromagnetic interactions have also been treated by several authors on the basis of different assumptions about meson fields and nuclear forces. It is, therefore, not likely that the present paper will contain any concrete result not already known to the physicists acquainted with the problems concerned. The interest it may nevertheless offer would rather lie in the more methodical aspects of the question, as viewed according to the general lines recalled above. By reason of this circumstance, we have decided also to include, when necessary, already well-known developments in an endeavour to present a rounded-off account of the questions treated.

Special care has been devoted to the more formal side of the theory, such as the systematic use and extension of the "symbolical space" algorithm already introduced in our first paper [1]. Since this system of representation might at first sight appear rather cumbrous when applied to charged meson fields, —in contrast to the more common representation of these fields by complex operators,—a few words on this subject will perhaps not be out of place here. An examination of the calculations developed in the Appendix with the help of this formalism will indeed show that they are not any lengthier or more intricate than those in which complex field variables are used. In most cases, we have even applied, for the sake of a more direct expression of the physical meaning of the quantities considered, a three-dimensional vector notation in preference to a four- or a five-dimensional tensor notation. This requires, it is true, some practice of vector analysis which otherwise would not be needed and is, of course, not so appropriate to questions involving space and time variables in a symmetrical way (an example of such a case is afforded by the calculation of the magnetic moment of free mesons; cf. p. 42); still, the symbolic space representation of the charged fields proves very convenient even then.

In the first part of the paper, we recall the general method, applicable to any field theory, of taking into account the interaction with the electromagnetic field; on this occasion, we adapt the general formulae to the symbolical space representation, using Hermitian field variables. In the last section of this part, the interaction with a slowly varying external field is more especially considered and the multipole moments suited to the treatment of such a case—in fact, only the first few ones: electric dipole, and quadrupole, and magnetic dipole moments—are introduced. The second part of the paper is chiefly concerned with the application of the canonical transformation separating the static meson fields to the interaction of a system of nucleons and mesons with the electromagnetic field. This amounts to deriving the expression of the charge and current density operators in terms of the transformed variables; from these, the expressions of the multipole moments just mentioned are then easily obtained. The resulting formulae for all such electromagnetic quantities contain, besides the usual terms corresponding to free nucleons and free mesons, and others depending on both nucleon and meson variables, also terms of the so-called "exchange" type, *i. e.* terms which depend only on the nucleon variables in the form of a sum of expressions involving exchange of proton and neutron state between pairs of nucleons; the occurrence of such exchange terms is a typical feature of field theory. There appear also, of course, further contributions of a singular character, which have to be rejected according to the above-mentioned general criterium; it is noteworthy, however, that a singular contribution corresponding to the anomalous magnetic moment of the nucleon may still be retained as being of a different origin, *viz.* due to the fact that a material point model is adopted for the nucleons*. It is clear that the method of canonical transformation here followed is superior to the usual perturbation method, above all in that it just permits easily to trace, so to say, the origin of the different terms occurring in any formula and so to facilitate their physical interpretation, especially in connexion with our general criterium. Furthermore, it also offers the practical advantage of requiring calculations much simpler and easier to survey than the perturbation theory.

As regards the application of the general scheme so obtained to special problems, we confine ourselves to some brief indications, since a more detailed treatment will be found in other publications. In this respect, we have already mentioned the investigation of the photo-effect of the deuteron by A. Pais [7]; we wish also to call attention to a forthcoming paper by J. Serpe which, among others, will contain a detailed comparison of the canonical transformation and perturbation methods and a discussion of two singular effects (spreading of electric charge around a nucleon and anomalous magnetic moment of nucleons) providing appropriate examples of the application of our criterium. We should like here to mention that results of calculations by these authors, kindly put by them at our disposal, have been very valuable for the final redaction of this paper.

Empirical evidence, especially the latest information on cosmic-ray mesons, may be considered on the whole to support the mixed theory [2]. Still, in conclusion, we wish to emphasize once more its very serious limitations, already explicitly stated in our previous paper ([1], especially pp. 51—52). In particular, it is out of question that such problems as the scattering of fast mesons [8] could fall within the scope of this or any other form of meson theory as long as the deep-lying problems connected with the universal limiting length (denoted by $r_0$ in [1]) are not brought nearer to their solution.

* In an analogous way, the static self-energy of a point nucleon may be considered a contribution to its mass.

# PART I.

## Electromagnetic properties of arbitrary fields.

### § 1. Gauge-invariance of Lagrangian.

Let us consider any field representing "charged" particles, *i. e.* particles capable of interaction with the electromagnetic field. If we assume that the force on such a charged particle due to the electromagnetic field is given by the familiar Lorentz expression, we may, according to an interesting theorem pointed out by RACAH [9], conclude that there exists a Lagrangian from which this force can be derived. In fact, to be derivable from a Lagrangian, the expressions of the force components have to satisfy a set of necessary and sufficient conditions established by HELMHOLTZ [10]; and RACAH has shown that, if we also assume the force to be independent of the acceleration of the particle, these conditions just reduce to stating that it takes the form of the Lorentz expression.

We may thus start from a Lagrangian density

$$\mathcal{L}\left(Q_\omega,\ Q_{\omega|i};\ Q^\dagger_\omega,\ Q^\dagger_{\omega|i};\ A_l,\ A_{l|i}\right)$$

depending on the variables $Q_\omega$, $Q^\dagger_\omega$, $A_l$ and their derivatives $F_{|i} = \dfrac{\partial F}{\partial x^i}$ $(x^i = x, y, z, ct)$; while the $A_l$ denote the components of the electromagnetic potential (vector potential $\vec{A} = A_1, A_2, A_3 = A^1, A^2, A^3$; scalar potential $B = A^4 = -A_4$), the $Q_\omega$ represent the quantized variables defining the various types of charged particles considered, $Q^\dagger_\omega$ their Hermitian conjugates, numbered in an arbitrary way by the index $\omega$. As is well-known [11]*,

* It will be noticed that the following analysis is entirely analogous to that used for the derivation of the energy-momentum tensor from the invariance of the Lagrangian for arbitrary coordinate transformations [12]. The possibility of extending this method to any group, and in particular to the gauge-transformation group, has been pointed out to us by Dr. J. PODOLANSKI.

the form of this Lagrangian giving rise to the required expression of the Lorentz force may simply be obtained by postulating the invariance of the Lagrangian for the so-called gauge-transformations of the form

$$\left\{\begin{aligned} Q'_\omega &= e^{\frac{i e_\omega \alpha}{\hbar c}} Q_\omega \\ A'_l &= A_l + \alpha_{|l}, \end{aligned}\right. \tag{1}$$

in which $\alpha$ is an arbitrary $c$-number function and the constants $e_\omega$, referring to the different kinds of particles considered, are to be identified with the charges of these particles. In fact, the gauge-invariance of any functional, such as $\mathscr{L}$, may be expressed in the form*

$$\delta \mathscr{L} \equiv \sum_\omega \left\{ \frac{\delta \mathscr{L}}{\delta Q_\omega} \delta Q_\omega + \frac{\partial}{\partial x^i}\left(\frac{\partial \mathscr{L}}{\partial Q_{\omega|i}} \delta Q_\omega\right) + \text{conj.}\right\}$$
$$+ \frac{\delta \mathscr{L}}{\delta A_i} \delta A_i + \frac{\partial}{\partial x^j}\left(\frac{\partial \mathscr{L}}{\partial A_{i|j}} \delta A_i\right) = 0,$$

where $\dfrac{\delta \mathscr{L}}{\delta Q}$ represents the "variational derivative" $\dfrac{\partial \mathscr{L}}{\partial Q} - \dfrac{\partial}{\partial x^i}\left(\dfrac{\partial \mathscr{L}}{\partial Q_{|i}}\right)$. Putting

$$s^i = \frac{1}{i} \sum_\omega \frac{e_\omega}{\hbar c} \left[\frac{\partial \mathscr{L}}{\partial Q_{\omega|i}} Q_\omega - Q^\dagger_\omega \frac{\partial \mathscr{L}}{\partial Q^\dagger_{\omega|i}}\right], \tag{2}$$

we then get, for an infinitesimal value of the parameter $\alpha$,

$$\left.\begin{aligned} \delta \mathscr{L} \equiv \alpha \left\{ \sum_\omega i \frac{e_\omega}{\hbar c} \frac{\delta \mathscr{L}}{\delta Q_\omega} Q_\omega + \text{conj.} - \frac{\partial s^i}{\partial x^i} \right\} \\ + \alpha_{|i} \left\{ \frac{\delta \mathscr{L}}{\delta A_i} + \frac{\partial}{\partial x^j}\left(\frac{\partial \mathscr{L}}{\partial A_{i|j}}\right) - s^i \right\} + \alpha_{|i|j} \frac{\partial \mathscr{L}}{\partial A_{i|j}} = 0. \end{aligned}\right\} \tag{3}$$

In connexion with the possible non-commutability of $Q_\omega$ and $Q^\dagger_\omega$, it has here been assumed that in all terms of $\mathscr{L}$ any factor $Q^\dagger_\omega$ or $Q^\dagger_{\omega|i}$ is written on the left of any $Q_\omega$ or $Q_{\omega|i}$; if $Q_{\omega|4}$ is not commutable with $Q_\omega$, further restrictions must be imposed

* The abbreviation "conj." denotes the Hermitian conjugate of the expression preceding it.

on the form of the Lagrangian if we wish to write down a general expression such as (3) for $\delta \mathscr{L}$. We will make the assumption, fulfilled in all cases of actual interest, that the expression (3) is valid in terms of $q$-number variables also. We then derive from it the three following conditions for the Lagrangian density $\mathscr{L}$, *viz.*:

1° $\mathscr{L}$ depends on the $A_{i|j}$ only through the field components

$$F_{ij} = A_{j|i} - A_{i|j}; \tag{4}$$

2° we have therefore

$$\frac{\delta \mathscr{L}}{\delta A_i} = s^i + \frac{\partial}{\partial x^j}\left(\frac{\partial \mathscr{L}}{\partial F_{ij}}\right); \tag{5}$$

3° we have

$$\frac{\partial s^i}{\partial x^i} = i \sum_{\omega} \frac{e_\omega}{\hbar c} \frac{\delta \mathscr{L}}{\delta Q_\omega} Q_\omega + \text{conj.} \tag{6}$$

Now, the total Lagrangian density of a system of charged particles and electromagnetic field includes, besides the term hitherto considered which refers to the particles and their interaction with the electromagnetic field, a term

$$\mathscr{L}_f = -\frac{1}{4} F^{ij} F_{ij} \tag{7}$$

describing the field itself. We may then write the field equations for the material variables $Q_\omega$, $Q_\omega^\dagger$ in the form

$$\frac{\delta \mathscr{L}}{\delta Q_\omega} = 0, \qquad \frac{\delta \mathscr{L}}{\delta Q_\omega^\dagger} = 0 \tag{8}$$

and the electromagnetic field equations in the form

$$\frac{\delta \mathscr{L}_f}{\delta A_i} = -\frac{\delta \mathscr{L}}{\delta A_i}$$

or

$$\frac{\partial F^{ij}}{\partial x^j} = \frac{\delta \mathscr{L}}{\delta A_i}. \tag{9}$$

From the latter it follows that $\frac{\delta \mathscr{L}}{\delta A_i}$ is to be interpreted as the total charge-current density of the system, the general expression of which is thus given by (5). In this formula, $s^i$, given by (2), is the usual charge-current density, whereas the last term represents a polarization current density, as would arise from an electric and magnetic moment of the particles through which they would directly interact with the electromagnetic field; such direct interaction we shall, however, disregard and we shall, thus, in the following assume that $\mathscr{L}$ only depends on the electromagnetic potential. In any case, since the divergence of the polarization current density automatically vanishes, the conservation law

$$\frac{\partial}{\partial x^i}\frac{\delta \mathscr{L}}{\delta A_i} = 0 \quad \text{or} \quad \frac{\partial s^i}{\partial x^i} = 0 \tag{10}$$

follows from (6) in virtue of the field equations (8).

In order to derive the expression of $\mathscr{L}$ in terms of the $A_i$, we start from the condition (5) which may be written more simply

$$\frac{\partial \mathscr{L}}{\partial A_i} = s^i \tag{5'}$$

and we further require $\mathscr{L}(Q_\omega, Q_{\omega|i};\, Q^\dagger_\omega, Q^\dagger_{\omega|i};\, A_l)$ to reduce for $A_l \to 0$ to the form $\mathscr{L}^\circ$ of the Lagrangian density for the particles concerned in the absence of electromagnetic field. It is now easy to show that these two requirements fix the form of $\mathscr{L}$ to*

$$\mathscr{L} = \mathscr{L}^\circ\left(Q_\omega, Q_{\omega|i} - i\frac{e_\omega}{\hbar c}A_i Q_\omega;\; Q^\dagger_\omega, Q^\dagger_{\omega|i} + i\frac{e_\omega}{\hbar c}A_i Q^\dagger_\omega\right). \tag{11}$$

This expresses, of course, the well-known result that the Lagrangian density for charged particles in interaction with the electromagnetic field is obtained from the Lagrangian for no field by replacing the operator $\frac{\partial}{\partial x^i}$ by $\frac{\partial}{\partial x^i} - i\frac{e_\omega}{\hbar c}A_i$.

* In the first place, the function (11) satisfies the two conditions. On the other hand, if a function of $Q_\omega, Q_{\omega|i};\, Q^\dagger_\omega, Q^\dagger_{\omega|i};\, A_l$ satisfies (5′) and reduces to zero for $A_l \to 0$, it is readily seen that it must be identically zero.

## § 2. Transition to Hamiltonian and separation of Coulomb field [13].

Let $P_\omega = \dfrac{\partial \mathscr{L}}{\partial Q_{\omega|4}}$ be the momentum canonically conjugate to $Q_\omega$; if $\mathscr{L}$ is Hermitian (which we will assume), the momentum canonically conjugate to $Q_\omega^\dagger$ is then $P_\omega^\dagger$. From the gauge-invariance of $\mathscr{L}$ it follows immediately that a gauge-transformation (1) transforms the variables $P_\omega$ according to the equation

$$P'_\omega = e^{-i\frac{e_\omega}{\hbar c}\alpha} P_\omega .$$

Let further $\mathscr{H}_m^\circ\left(Q_\omega,\ \mathrm{grad}\ Q_\omega,\ P_\omega^\circ;\ Q_\omega^\dagger,\ \mathrm{grad}\ Q_\omega^\dagger,\ P_\omega^{\circ\dagger}\right)$ be the Hamiltonian function with no electromagnetic field present, defined in the usual way by the equation

$$\mathscr{H}_m^\circ = \sum_\omega P_\omega^\circ\, Q_{\omega|4} + \mathrm{conj.} - \mathscr{L}^\circ$$

from which the variables $Q_{\omega|4}$ are understood to be eliminated by means of the equations

$$P_\omega^\circ = \frac{\partial}{\partial Q_{\omega|4}}\ \mathscr{L}^\circ\left(Q_\omega,\ Q_{\omega|i};\ Q_\omega^\dagger,\ Q_{\omega|i}^\dagger\right).$$

The Hamiltonian with the field present, as derived from the Lagrangian (11), is then easily seen to be*

$$\mathscr{H}_m + \int B\varrho\, dv,$$

where

$$\varrho \equiv s^4 = \frac{1}{i}\sum_\omega \frac{e_\omega}{\hbar c}\left[P_\omega Q_\omega - Q_\omega^\dagger P_\omega^\dagger\right]$$

is the charge density and

$$\mathscr{H}_m = \mathscr{H}_m^\circ\left(Q_\omega,\ \left(\mathrm{grad} - \frac{ie_\omega}{\hbar c}\vec{A}\right) Q_\omega,\ P_\omega;\ \cdots\cdots\right) \tag{12}$$

the gauge-invariant part of the Hamiltonian. Similarly, the Hamiltonian of the pure electromagnetic field (electric field $\vec{E}$, magnetic field $\vec{H}$) derived from the Lagrangian (7) is

* $\int \cdots\cdots dv$ denotes a volume integration over the whole space.

$$\mathcal{H}_f - \int B \operatorname{div} \vec{E}\, dv,$$

with

$$\mathcal{H}_f = \frac{1}{2}\int (\vec{E}^2 + \vec{H}^2)\, dv; \tag{12'}$$

the momentum canonically conjugate to $\vec{A}$ is $-\vec{E}$, while that canonically conjugate to $B$ is identically zero. The Hamiltonian of the total system may therefore be written

$$\mathcal{H} = \mathcal{H}_f + \mathcal{H}_m - \int B\, \mathcal{C}\, dv, \tag{13}$$

with

$$\mathcal{C} = \operatorname{div} \vec{E} - \varrho. \tag{13'}$$

The Hamiltonian formalism must be completed, in the usual way, by the fundamental commutation rules and the accessory condition

$$\mathcal{C} = 0 \tag{14}$$

to be imposed on the wave-function describing the state of the system. While the variables $\vec{A}$ and $-\vec{E}$ satisfy the ordinary canonical commutation rules, the variable $B$ commutes with all other electromagnetic field quantities.

Let us now apply to the Hamiltonian $\mathcal{H}$ the canonical transformation corresponding to an arbitrary change of phase of the variables $Q_\omega$, $P_\omega$:

$$Q'_\omega = e^{i\frac{e_\omega}{\hbar c}\alpha} Q_\omega, \quad P'_\omega = e^{-i\frac{e_\omega}{\hbar c}\alpha} P_\omega, \tag{15}$$

whereby $\alpha$ may be any operator independent of $Q_\omega$, $P_\omega$ and $\vec{E}$ (but eventually containing the electromagnetic potentials); such a transformation may, in fact, be put into the form*

$$Q'_\omega = \mathcal{S}^{-1} Q_\omega\, \mathcal{S}, \quad P'_\omega = \mathcal{S}^{-1} P_\omega\, \mathcal{S}, \tag{15}$$

with

$$\mathcal{S} = e^{\frac{i}{\hbar c}\int \varrho\, \alpha\, dv}. \tag{16}$$

* This may readily be verified for tensor variables (satisfying canonical commutation rules) as well as for spinor variables (which obey commutation rules corresponding to the exclusion principle).

Expressed in the new (primed) variables, the part $\mathscr{H}_m$ of the Hamiltonian is given by

$$\mathscr{H}_{m\,(\text{new var.})} = \mathscr{S}' \mathscr{H}'_m \mathscr{S}'^{-1},$$

where $\mathscr{H}'_m$, in accordance with a general convention made in this paper, denotes the same function of the new variables as the function $\mathscr{H}_m$ of the old variables. Now, the gauge-invariance of $\mathscr{H}_m$ may be stated in the form

$$\mathscr{H}_m(\vec{A}, \cdots\cdots) = \mathscr{S}^{-1} \mathscr{H}_m(\vec{A} + \operatorname{grad}\alpha, \cdots\cdots)\mathscr{S},$$

from which it follows, according to (12), that

$$\left.\begin{aligned} &\mathscr{H}_{m\,(\text{new var.})} = \\ &= \mathscr{H}^{\circ}_m\left(Q'_\omega, \left(\operatorname{grad} - i\frac{e_\omega}{\hbar c}\vec{A}' - i\frac{e_\omega}{\hbar c}\operatorname{grad}\alpha\right)Q'_\omega,\ P'_\omega;\ \cdots\cdots\right). \end{aligned}\right\} \quad (17)$$

As to the other terms of the Hamiltonian (13), they may be written, if we put

$$\mathscr{S}\vec{E}\,\mathscr{S}^{-1} = \vec{E} + \vec{E}_0 \qquad (18)$$

and

$$\mathscr{V} = \frac{1}{2}\int \vec{E}_0^2\, dv, \qquad (19)$$

in the form

$$\mathscr{H}_{f\,(\text{new var.})} = \mathscr{H}'_f + \frac{1}{2}\int\left(\vec{E}'\vec{E}'_0 + \vec{E}'_0\vec{E}'\right)dv + \mathscr{V}' \qquad (20)$$

and

$$-\int B\,\mathscr{C}_{(\text{new var.})}\,dv, \quad \text{with} \quad \mathscr{C}_{(\text{new var.})} = \operatorname{div}\vec{E}' + \operatorname{div}\vec{E}'_0 - \varrho'. \quad (21)$$

Now, we can choose $\alpha$ in such a way that the field $\vec{E}_0$ separated from the total field by the canonical transformation $\mathscr{S}$ be just the Coulomb field

$$\left\{\begin{aligned} &\vec{E}_0 = -\operatorname{grad}\int \varrho\,\varphi_0\,dv, \quad \varphi_0 = \frac{1}{4\pi r}; \\ &\operatorname{div}\vec{E}_0 = \varrho. \end{aligned}\right. \qquad (22)$$

We have only to take

$$\alpha = \int \operatorname{div} \vec{A} \cdot \varphi_0 \, dv, \tag{23}$$

whence

$$\mathcal{S} = e^{\frac{i}{\hbar c} \int \vec{E}_0 \cdot \vec{A} \, dv}. \tag{24}$$

The new field variable $\vec{E}'$ then represents the transverse part of the electric field; since, according to (23),

$$\operatorname{rot}(-\operatorname{grad} \alpha) = 0, \quad \operatorname{div}(-\operatorname{grad} \alpha) = -\int \operatorname{div} \vec{A} \cdot \triangle \varphi_0 \, dv = \operatorname{div} \vec{A},$$

we also see that $-\operatorname{grad} \alpha$ may be interpreted as the longitudinal part of the vector-potential $\vec{A}$, so that in the new Hamiltonian (17) only the transverse part

$$\vec{A}_\perp = \vec{A} + \operatorname{grad} \alpha$$

(or rather $\vec{A}'_\perp$) occurs, and this variable $\vec{A}'_\perp$ also defines the magnetic field by $\vec{H}' = \operatorname{rot} \vec{A}'_\perp$. On account of the condition (14), the value of $B$ in the Hamiltonian (13) may be chosen arbitrarily; taking for it the Coulomb potential $\int \varrho \varphi_0 \, dv$, one causes the terms

$$\frac{1}{2} \int \left( \vec{E}' \vec{E}_0' + \vec{E}_0' \vec{E}' \right) dv = \int \vec{E}' \vec{E}_0' \, dv$$

and

$$-\int B \, \mathcal{C}_{(\text{new var.})} \, dv = -\int B \operatorname{div} \vec{E}' \, dv$$

to cancel each other. Finally, the term $\mathcal{V}'$ simply represents (in the primed variables) the Coulomb energy

$$\mathcal{V} = \frac{1}{2} \int \varrho\left(\vec{x}\right) \varrho\left(\vec{x}'\right) \varphi_0 \left( \left| \vec{x} - \vec{x}' \right| \right) dv \, dv',$$

so that we get a Hamiltonian

$$\left.\begin{aligned} &\mathcal{H}_{(\text{new var.})} = \\ &= \mathcal{H}_m^\circ \left( Q_\omega', \left( \operatorname{grad} - \frac{i e_\omega}{\hbar c} \vec{A}_\perp' \right) Q_\omega', P_\omega'; \cdots\cdots \right) + \mathcal{H}_f' + \mathcal{V}', \end{aligned}\right\} \tag{25}$$

from which the longitudinal parts of electric field and vector potential are entirely eliminated. Also the accessory condition

$$\mathcal{C}_{(\text{new var.})} \equiv \operatorname{div} \vec{E}' = 0 \tag{26}$$

does no more contain the longitudinal field; but it must not be forgotten that $\vec{A}'$, and not $\vec{A}'_{\perp}$, is canonically conjugate to $-\vec{E}'$.

## § 3. Hermitian variables.

Chiefly when dealing with meson fields it may be advantageous to use Hermitian variables for the description of the charged particles. As such, we may conveniently take

$$Q_1 = \frac{1}{\sqrt{2}}(Q + Q^{\dagger}), \quad Q_2 = \frac{1}{i\sqrt{2}}(Q - Q^{\dagger}),$$

so that

$$Q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2). \tag{27}$$

Considering these formally as "components" of a "vector" $\boldsymbol{Q}$ along the directions of orthogonal axes $\mathbf{1}$ and $\mathbf{2}$, we may interpret the phase transformation $Q' = e^{i\alpha} Q$ as a "rotation" of angle $\alpha$ about the symbolical axis $\mathbf{3}$, perpendicular to $\mathbf{1}$ and $\mathbf{2}$. Any (Hermitian) "component" $Q_{\mathbf{3}}$ in the direction of the axis $\mathbf{3}$ can be assumed to represent a neutral particle of the same kind. Taking the vector potential (and all electromagnetic field components) along the axis $\mathbf{3}$:

$$\vec{\boldsymbol{A}} \equiv \left(0, 0, \vec{A}\right),$$

we may then express the gauge-invariant derivatives

$$Q_{|l} - \frac{ie}{\hbar c} A_l Q, \quad Q^{\dagger}_{|l} + \frac{ie}{\hbar c} A_l Q^{\dagger},$$

through which the interaction with the electromagnetic field is introduced into the Hamiltonian, in the form

$$\boldsymbol{Q}_{|l} - \frac{e}{\hbar c} \boldsymbol{A}_l \wedge \boldsymbol{Q},$$

the sign $\wedge$ denoting a vector product in symbolical space. Calling finally $\boldsymbol{P}$ the momenta canonically conjugate to $\boldsymbol{Q}$, we have

$$P = \frac{1}{\sqrt{2}} (P_1 - i P_2),$$

and, since the Lagrangian density $\mathscr{L}$ has been assumed to be Hermitian, $P_1$ and $P_2$ are also Hermitian.

For the electric charge density, we have, according to (2), the general formula*

$$\left.\begin{aligned} \varrho &= \frac{1}{i} \sum_{\omega} \frac{e_\omega}{\hbar c} \left(P_\omega Q_\omega - Q^\dagger_\omega P^\dagger_\omega\right) \\ &= \sum_{\omega} \frac{e_\omega}{\hbar c} \left\{ \frac{1}{2} \left[\boldsymbol{P}_\omega \wedge \boldsymbol{Q}_\omega - \boldsymbol{Q}_\omega \wedge \boldsymbol{P}_\omega\right]_3 + \right. \\ &\quad \left. + \frac{1}{2i} \left[\left[P_{\omega 1}, Q_{\omega 1}\right]_- + \left[P_{\omega 2}, Q_{\omega 2}\right]_-\right] \right\}; \end{aligned}\right\} \tag{28}$$

for the current density $\vec{I}$, analogous formulae hold, $\boldsymbol{P}_\omega$ being replaced by $\frac{\partial \mathscr{L}}{\partial \boldsymbol{Q}_{\omega|i}}$ $(i = 1, 2, 3)$. In order further to discuss these expressions, we must specify the commutation rules assumed for the $\boldsymbol{P}_\omega$ and $\boldsymbol{Q}_\omega$. Let us first consider a field, describing particles of charge $e$, which obeys the canonical commutation rules

$$\left[P_{\omega i}(\vec{x}), Q_{\omega' i'}(\vec{x}')\right]_- = \frac{\hbar c}{i} \delta\left(\vec{x} - \vec{x}'\right) \delta_{\omega\omega'} \delta_{i i'}, \tag{29}$$

the index $\omega$ referring to the different components of the field. The last term in $\varrho$, which occurs for every given state of the field, then would represent an infinite additional charge density of this state; such a term, however, can easily be avoided by a slight modification of the Lagrangian density $\mathscr{L}$; we have only to replace $\mathscr{L}$ by half the sum of $\mathscr{L}$ itself and the expression $\mathscr{L}'$ obtained from it by reversing the order of all factors** $Q$, $Q_{|i}$, $Q^\dagger$, $Q^\dagger_{|i}$. We may, therefore, disregard the infinite "zero-point" charge density altogether and write, in this case,

* We use the notation $[A, B]_\pm = AB \pm BA$.

** In fact, if we repeat for $\mathscr{L}'$ the considerations of § 1, we find for $\delta \mathscr{L}'$ and, consequently, for the charge-current density expressions in which the order of factors is automatically reversed. We may further assume that $\frac{\partial \mathscr{L}'}{\partial Q_{\omega|4}} = \frac{\partial \mathscr{L}}{\partial Q_{\omega|4}}$, a condition which is fulfilled in all cases of actual interest. We then get

$$\left.\begin{aligned} \varrho &= \frac{e}{\hbar c} \sum_{\omega} \left(\boldsymbol{P}_{\omega} \wedge \boldsymbol{Q}_{\omega}\right)_{\mathbf{3}} \\ I_i &= \frac{e}{\hbar c} \sum_{\omega} \left(\frac{\partial \mathscr{L}}{\partial \boldsymbol{Q}_{\omega | i}} \wedge \boldsymbol{Q}_{\omega}\right)_{\mathbf{3}}. \end{aligned}\right\} \tag{30}$$

As an example of a field of this type, we may of course mention the meson field to be extensively treated in the following.

For the sake of completeness, we shall still briefly examine the case of Dirac particles (electrons or nucleons) described by four component wave-functions satisfying the commutation rules which correspond to the exclusion principle. The decomposition (27) of such variables into Hermitian constituents has been especially discussed, in connexion with the theory of electron pairs, by MAJORANA [14]; it will thus suffice here to write down the main formulae of this theory without entering into details as to their interpretation. The commutation rules may be stated as follows:

$$\left[Q_{\omega\mathbf{1}}(\vec{x}),\, Q_{\omega'\mathbf{1}}(\vec{x}')\right]_+ = \left[Q_{\omega\mathbf{2}}(\vec{x}),\, Q_{\omega'\mathbf{2}}(\vec{x}')\right]_+ = \delta(\vec{x}-\vec{x}')\,\delta_{\omega\omega'},$$

$$\text{all } Q_{\omega\mathbf{1}} \text{ anticommute with all } Q_{\omega\mathbf{2}};$$

the indices $\omega, \omega' (= 1, 2, 3, 4)$ refer to the components of the (spinor) wave-function; we may in the usual way consider the complex wave-function $Q$ or its Hermitian constituents $\boldsymbol{Q}$ as matrices, with respect to $\omega$, of 4 rows and 1 column. For any Hermitian operator $\mathscr{O}$, we may then write*

---

$$\varrho = \frac{1}{i} \sum_{\omega} \frac{e_\omega}{\hbar c} \left\{\left(P_\omega Q_\omega - Q^\dagger_\omega P^\dagger_\omega\right) + \left(Q_\omega P_\omega - P^\dagger_\omega Q^\dagger_\omega\right)\right\},$$

which eliminates the zero-point charge density.

In this special case, it would also be possible (cf. [15]) to define the charge density by the expression

$$\varrho = \frac{1}{i} \sum_{\omega} \frac{e_\omega}{\hbar c} \left(P_\omega Q_\omega - P^\dagger_\omega Q^\dagger_\omega\right),$$

which does not contain a zero-point density either; this form, however, has the disadvantage of not being Hermitian by itself, but only on account of the commutation rules.

* $\tilde{A}$ denotes the transposed of $A$, and $A^*$ its complex conjugate; one has $A^\dagger = \tilde{A}^*$.

$$\int Q^{\dagger} \mathscr{O} Q\, dv =$$

$$= \frac{1}{2}\int \tilde{Q}_1 \mathscr{O} Q_1\, dv + \frac{1}{2}\int \tilde{Q}_2 \mathscr{O} Q_2\, dv + \frac{i}{2}\int \tilde{Q}_1 (\mathscr{O} + \mathscr{O}^*) Q_2\, dv\,;$$

in particular, for a purely imaginary operator, the last term disappears. Choosing for instance, with MAJORANA, a representation of the Dirac matrices for which the Hamiltonian $\mathscr{H}^{\circ}_{m}$ is purely imaginary, we get for the total energy of a system of charged (and eventually neutral) Dirac particles the expression

$$\frac{1}{2}\int \tilde{\boldsymbol{Q}} \mathscr{H}^{\circ}_{m} \boldsymbol{Q}\, dv$$

used by MAJORANA. Since

$$P_1 = \frac{1}{2}\hbar c\, \tilde{Q}_2, \qquad P_2 = -\frac{1}{2}\hbar c\, \tilde{Q}_1,$$

we may now write

$$\sum_{\omega=1}^{4} [P_{\omega 1}, Q_{\omega 1}]_- = \sum_{\omega=1}^{4} [P_{\omega 2}, Q_{\omega 2}]_- = -\frac{1}{2}\hbar c (\tilde{\boldsymbol{Q}} \wedge \boldsymbol{Q})_3\,;$$

the last term of (28) now gives

$$\varrho = ie(\tilde{\boldsymbol{Q}} \wedge \boldsymbol{Q})_3,$$

while the first would, on account of the commutation rules, contribute an infinite "zero-point charge density"; but this may again be avoided by taking instead of $\mathscr{L}$ half the difference of $\mathscr{L}$ and $\mathscr{L}'$. For the current density, we find

$$\vec{I} = ie\left(\tilde{\boldsymbol{Q}}\, \vec{\alpha} \wedge \boldsymbol{Q}\right)_3,$$

$\vec{\alpha}$ denoting the well-known velocity matrices of the Dirac electron theory. All these formulae coincide, of course, with those of MAJORANA.

## § 4. System in slowly varying external field.

An important class of problems is concerned with the behaviour of a system of charged particles under the action of an external electromagnetic field. We have then to introduce into the Hamiltonian of the system additional terms expressing the energy of interaction between this field and the particles considered. Let us in this section denote by $\vec{A}$, $B$; $\vec{E}$, $\vec{H}$ the variables of the external field. On account of (5′) we may write, for the part of the interaction energy depending linearly on $\vec{A}$,

$$-\int \vec{A}\,\vec{I}\,dv + \int B\,\varrho\,dv, \tag{31}$$

where we must take for the charge and current densities $\vec{I}$, $\varrho$ their expressions in absence of any electromagnetic field (for $\vec{A} = 0$): we shall neglect any contribution of higher order in $\vec{A}$. The expression (31) is not modified by the canonical transformation $\mathcal{S}$ effecting the separation of the Coulomb field, so that, after this separation has been performed, it keeps the same form in terms of the new variables; we may here without danger of confusion omit the primes indicating functions of the new variables. More particularly, we shall consider a "slowly varying" field, *i. e.* assume that the relative variations of the field components over distances of the order of the dimensions of the system and times of the order of the proper periods connected with the system are small compared with unity. In such a case, we may put the expression of the interaction of the system with the external field into a more convenient form depending on the values of the electromagnetic field and its derivatives at some arbitrarily chosen point $O$ and on the successive $2^n$-pole moments of the system with respect to this point. The properties of the system to any desired approximation are then easily derived from the expressions of such $2^n$-pole moments, so that the problem is essentially reduced to the calculation of these operators.

We shall now carry out the transformation of the interaction operator up to the second order of approximation. If we further assume that the velocities of the particles are small compared with the velocity of light, *i. e.* that the dimensions of the system are small compared with the proper wave-lengths, this approximation involves

$$\left.\begin{array}{ll}\text{the total charge} & e = \int \varrho\, dv, \\ \text{the electric dipole moment} & \vec{P} = \int \varrho\, \vec{x}\, dv, \\ \text{the magnetic dipole moment} & \vec{M} = \frac{1}{2}\int \vec{x} \wedge \vec{I}\, dv, \\ \text{and} & \\ \text{the electric quadrupole moment*} & Q^{ik} = \frac{1}{2}\int \varrho\, x^i x^k\, dv \end{array}\right\} \quad (32)$$

of the system, the vector $\vec{x}$ being taken from $O$ as origin. For this purpose, we start from the expansions

$$\left\{\begin{array}{l} B = B_O + \vec{x}\,\mathrm{grad}_O B_O + \frac{1}{2}\vec{x}\left(\vec{x}\,\mathrm{grad}_O\right)\mathrm{grad}_O B_O + R \\ \vec{A} = \vec{A}_O + \left(\vec{x}\,\mathrm{grad}_O\right)\vec{A}_O + \vec{R}', \end{array}\right.$$

the residuals $R$, $\vec{R}'$ containing higher derivatives of the potentials with respect to the point $O$. Remembering further that**

$$\int \vec{I}\, dv = \dot{\vec{P}},$$

we therefore get***

$$\left\{\begin{array}{l} \int B \varrho\, dv = e B_O + \vec{P}\,\mathrm{grad}_O B_O + (Q\,\mathrm{grad}_O)\,\mathrm{grad}_O B_O + R'' \\ \int \vec{A}\vec{I}\, dv = \dot{\vec{P}}\vec{A}_O + \int \vec{I}\left(\vec{x}\,\mathrm{grad}_O\right)\vec{A}_O\, dv + \vec{R}''', \end{array}\right.$$

the residuals $R''$, $\vec{R}'''$ being of higher order of approximation than the second. Now,

$$\begin{aligned}\left(\vec{x}\,\mathrm{grad}_O\right)\vec{A}_O &= -\left(\vec{A}_O\,\mathrm{grad}_O\right)\vec{x} + \mathrm{grad}_O\left(\vec{A}_O\vec{x}\right) - \vec{x} \wedge \mathrm{rot}_O \vec{A}_O \\ &= \vec{A}_O + \mathrm{grad}_O\left(\vec{A}_O\vec{x}\right) - \vec{x} \wedge \vec{H}_O;\end{aligned}$$

and

* See the footnote on p. 21.

** We use the notation $\dot{A} = \frac{\partial A}{\partial x^4}\left(\text{or } \frac{dA}{dx^4}\right)$.

*** If $Q$ is a tensor and $\vec{u}$ a vector, $Q\vec{u}$ represents the vector with components $\sum_k Q^{ik} u_k$.

$$\vec{I}\left(\vec{x}\,\mathrm{grad}_O\right)\vec{A}_O = \vec{I}\vec{A}_O + \vec{x}\left(\vec{I}\,\mathrm{grad}_O\right)\vec{A}_O + \vec{A}_O\left(\vec{I}\,\mathrm{grad}_O\right)\vec{x} - \left(\vec{I}\wedge\vec{x}\right)\vec{H}_O$$
$$= \vec{x}\left(\vec{I}\,\mathrm{grad}_O\right)\vec{A}_O + \left(\vec{x}\wedge\vec{I}\right)\vec{H}_O;$$

on the other hand,

$$\vec{I}\left(\vec{x}\,\mathrm{grad}_O\right)\vec{A}_O + \vec{x}\left(\vec{I}\,\mathrm{grad}_O\right)\vec{A}_O = \vec{I}\,\mathrm{grad}\left\{\vec{x}\left(\vec{x}\,\mathrm{grad}_O\right)\vec{A}_O\right\}.$$

We may therefore write

$$\vec{I}\left(\vec{x}\,\mathrm{grad}_O\right)\vec{A}_O = \frac{1}{2}\left(\vec{x}\wedge\vec{I}\right)\vec{H}_O + \frac{1}{2}\vec{I}\,\mathrm{grad}\left\{\vec{x}\left(\vec{x}\,\mathrm{grad}_O\right)\vec{A}_O\right\},$$

whence

$$\int\vec{I}\left(\vec{x}\,\mathrm{grad}_O\right)\vec{A}_O\,dv = \vec{M}\vec{H}_O - \frac{1}{2}\int\mathrm{div}\,\vec{I}\cdot\vec{x}\left(\vec{x}\,\mathrm{grad}_O\right)\vec{A}_O\,dv$$
$$= \vec{M}\vec{H}_O + (\dot{Q}\,\mathrm{grad}_O)\,\vec{A}_O.$$

With this result, we thus get

$$-\int\vec{A}\vec{I}\,dv = -\dot{\vec{P}}\vec{A}_O - \vec{M}\vec{H}_O - (\dot{Q}\,\mathrm{grad}_O)\,\vec{A}_O - \vec{R}'''$$
$$= \vec{P}\,\dot{\vec{A}}_O - \vec{M}\vec{H}_O + (Q\,\mathrm{grad}_O)\,\dot{\vec{A}}_O$$
$$-\frac{d}{c\,dt}\left[\vec{P}\vec{A}_O + (Q\,\mathrm{grad}_O)\,\vec{A}_O\right] - \vec{R}'''$$

and, finally,

$$\left.\begin{aligned} &-\int\vec{A}\vec{I}\,dv + \int B\varrho\,dv = \\ &= eB_O - \vec{P}\vec{E}_O - \vec{M}\vec{H}_O - (Q\,\mathrm{grad}_O)\,\vec{E}_O \\ &-\frac{d}{c\,dt}\left[\vec{P}\vec{A}_O + (Q\,\mathrm{grad}_O)\,\vec{A}_O\right] + \cdots\cdots, \end{aligned}\right\} \quad (33)$$

* It should be pointed out that the usual (and, from a systematic point of view, more rational) definition of the quadrupole moment would be expressed in our notation by $Q^{ik} - \frac{1}{3}\delta^{ik}\sum_l Q^{ll}$. This makes, however, no difference for the interaction term $-(Q\,\mathrm{grad}_O)\,\vec{E}_O$ since, for an external field, $\mathrm{div}_O\,\vec{E}_O = 0$.

the residual terms being of higher order. All terms but the last in formula (33) are familiar expressions, the interpretation of which it is superfluous to recall*. As regards the last term, it follows from its being the time derivative of an operator that it does not give any contribution either to the mean value of the interaction energy in stationary states of the system or to the matrix elements corresponding to transitions between states of the total system with the same energy.

* See the footnote on p. 21.

# PART II.

## Electromagnetic properties of a system of nucleons and mesons.

In our previous paper [1]*, arguments have been developed for assuming that nucleons produce two independent kinds of meson fields, *viz.* vector and pseudoscalar fields, each consisting of both (positively and negatively) charged and neutral mesons. From the standpoint of this "mixed" theory, we shall now investigate the interaction of a system of nucleons and of such meson fields with the electromagnetic field. In the first place, we shall derive, by a direct application of the general considerations of Part I of the present paper, the expression of the Hamiltonian function of the total system. To this Hamiltonian we shall then apply the canonical transformation effecting, as explained in NF, the separation of the static meson fields produced by the nucleons and we shall briefly discuss the different electromagnetic interaction terms obtained in this way; we shall especially fix our attention on the interaction with an external electromagnetic field.

### § 1. **Hamiltonian of the total system.**

The expression of the total Hamiltonian, after separation of the Coulomb field, may immediately be obtained from formula (25) of Part I (in which we may, from now on, omit the primes denoting the "new" variables). For the description of the nucleons and meson fields, we shall use the notations of NF, to which we beg the reader to refer for the explanation of their meaning; to begin with, we have to do with the variables

* In the following, this paper will be referred to as NF, its formula (n) as NF (n).

originally introduced there before the separation of the static parts of the meson fields. Treating the nucleons as Dirac particles, *i. e.* attributing to them the Hamiltonian $\mathscr{H}_k$ defined by formula NF (6), we get, according to formulae (25) and (31) of Part I, an additional term of electromagnetic interaction

$$-\int \vec{I}_{\text{nucl}} \cdot \vec{A}\, dv,$$

where $\vec{A}$ represents the sum of the transversal vector-potential $\vec{A}_{\perp}$ of the field considered a part of the system and of the vector-potential $\vec{A}_{\text{ex}}$ of any "external" field eventually acting on it. The current-density $\vec{I}_{\text{nucl}}$ of the nucleons may be expressed, with our choice of variables, by

$$\vec{I}_{\text{nucl}} = e \sum_i \frac{1-\tau_3^{(i)}}{2} \vec{\alpha}^{(i)} \delta\left(\vec{x} - \vec{x}^{(i)}\right). \tag{1}$$

If there is also an external scalar potential $B_{\text{ex}}$, a further term

$$\int \varrho_{\text{nucl}} \cdot B_{\text{ex}}\, dv$$

arises, in which the charge density $\varrho_{\text{nucl}}$ is given by

$$\varrho_{\text{nucl}} = e \sum_i \frac{1-\tau_3^{(i)}}{2} \delta\left(\vec{x} - \vec{x}^{(i)}\right). \tag{2}$$

As regards the meson fields, let us start from the Lagrangian density with no electromagnetic field present (which had not been stated explicitly in NF):

$$\left.\begin{aligned} \mathscr{L}^{\circ} = & -\frac{1}{2}\left\{\kappa^2\left(\vec{\boldsymbol{U}}^2 - \boldsymbol{V}^2\right) + \vec{\boldsymbol{G}}^2 - \vec{\boldsymbol{F}}^2\right\} + \vec{\boldsymbol{M}}\,\vec{\boldsymbol{U}} - \boldsymbol{N}\,\boldsymbol{V} \\ & -\frac{1}{2}\left\{\kappa^2\,\boldsymbol{\Psi}^2 \qquad + \vec{\boldsymbol{\Gamma}}^2 - \boldsymbol{\Phi}^2\right\} + \boldsymbol{R}\,\boldsymbol{\Psi} \qquad + \frac{1}{2}\left(\vec{\boldsymbol{P}}^2 - \boldsymbol{Q}^2\right), \end{aligned}\right\} \tag{3}$$

in which the variables $\boldsymbol{Q}_{\omega}$ are $\vec{\boldsymbol{U}}$, $\boldsymbol{V}$ and $\boldsymbol{\Psi}$, while $\vec{\boldsymbol{G}}$, $\vec{\boldsymbol{F}}$, $\vec{\boldsymbol{\Gamma}}$, $\boldsymbol{\Phi}$ are defined by the second formula NF (2), the first NF (1), NF (22), and the first NF (21), respectively. Following the procedure of Part I, we first see that $-\vec{\boldsymbol{F}}$ and $\boldsymbol{\Phi}$ are canonically conjugate to $\vec{\boldsymbol{U}}$ and $\boldsymbol{\Psi}$, respectively, while the variable $\boldsymbol{V}$, the conjugate momentum of which identically vanishes, must in this case be regarded as defined in terms of the other variables by the accessory Lagrangian equation, generalizing the first NF (2),

$$\kappa^2 \boldsymbol{V} = -\operatorname{div} \vec{\boldsymbol{F}} + \boldsymbol{N} + \frac{e}{\hbar c} \vec{\boldsymbol{A}} \wedge \vec{\boldsymbol{F}}. \tag{3'}$$

Let us now introduce the notations*

$$\left\{\begin{aligned} \hat{\boldsymbol{V}} &= \boldsymbol{V} + \kappa^{-2} \frac{e}{\hbar c} \vec{\boldsymbol{A}} \wedge \vec{\boldsymbol{F}} \\ \hat{\vec{\boldsymbol{G}}} &= \vec{\boldsymbol{G}} - \frac{e}{\hbar c} \vec{\boldsymbol{A}} \mathbin{\hat{\wedge}} \vec{\boldsymbol{U}} \\ \hat{\vec{\boldsymbol{\Gamma}}} &= \vec{\boldsymbol{\Gamma}} + \frac{e}{\hbar c} \vec{\boldsymbol{A}} \wedge \boldsymbol{\Psi}, \end{aligned}\right. \tag{4}$$

the quantities $\boldsymbol{V}$, $\vec{\boldsymbol{G}}$, $\vec{\boldsymbol{\Gamma}}$ being defined by NF (2) and NF (22); denote further by $\hat{A}$ the function $A$ of $\boldsymbol{V}$, $\vec{\boldsymbol{G}}$, $\vec{\boldsymbol{\Gamma}}$, in which these quantities have been replaced by $\hat{\boldsymbol{V}}$, $\hat{\vec{\boldsymbol{G}}}$, $\hat{\vec{\boldsymbol{\Gamma}}}$, respectively. Since the accessory equation (3′) amounts to saying that the variable $\boldsymbol{V}$ has to be replaced by the quantity $\hat{\boldsymbol{V}}$ just introduced by the first formula (4), we may with this notation, according to the general results of Part I, write the Hamiltonian of the meson fields, including the interaction with the electromagnetic field, in the form

$$\hat{\mathscr{H}}_F + \hat{\mathscr{H}}_\Phi + \int \varrho_{\mathrm{mes}} \cdot B_{\mathrm{ex}}\, dv;$$

in this formula, $\mathscr{H}_F$ and $\mathscr{H}_\Phi$ are defined by NF (7) and NF (26), respectively, while $\varrho_{\mathrm{mes}}$ denotes the meson charge density

$$\varrho_{\mathrm{mes}} = \frac{e}{\hbar c} \left\{ -\vec{\boldsymbol{F}} \wedge \vec{\boldsymbol{U}} + \boldsymbol{\Phi} \wedge \boldsymbol{\Psi} \right\}_3 \tag{5}$$

as given by formula (30) of Part I. For the meson current density we get

$$\hat{\vec{I}}_{\mathrm{mes}} = \frac{e}{\hbar c} \left\{ \hat{\vec{\boldsymbol{G}}} \mathbin{\hat{\wedge}} \vec{\boldsymbol{U}} - \vec{\boldsymbol{F}} \wedge \hat{\boldsymbol{V}} + \hat{\vec{\boldsymbol{\Gamma}}} \wedge \boldsymbol{\Psi} \right\}_3. \tag{6}$$

* The sign $\hat{\wedge}$ denotes a vector product in ordinary as well as in symbolical space.

This expression thus depends on the vector potential $\vec{A}$; for $\vec{A} = 0$, it goes over into

$$\vec{I}_{\text{mes}} = \frac{e}{\hbar c}\left\{\vec{\boldsymbol{G}} \wedge \vec{\boldsymbol{U}} - \vec{\boldsymbol{F}} \wedge \boldsymbol{V} + \vec{\boldsymbol{\Gamma}} \wedge \boldsymbol{\Psi}\right\}_3. \tag{7}$$

Accordingly, the part of the above Hamiltonian representing the interaction energy between meson fields and electromagnetic field consists of a linear term

$$-\int \vec{I}_{\text{mes}} \cdot \vec{A}\, dv$$

and a quadratic term

$$+\frac{1}{2}\left(\frac{e}{\hbar c}\right)^2 \int dv \left\{\left(\vec{\boldsymbol{A}} \wedge \vec{\boldsymbol{U}}\right)^2 + \left(\vec{\boldsymbol{A}} \wedge \boldsymbol{\Psi}\right)^2 + \kappa^{-2}\left(\vec{\boldsymbol{A}} \wedge \vec{\boldsymbol{F}}\right)^2\right\}. \tag{8}$$

The latter we will in the following neglect altogether; we shall, therefore, only be concerned with the expression $\vec{I}_{\text{nucl}} + \vec{I}_{\text{mes}}$, keeping in mind, however, that it does not represent the total current density when an electromagnetic field is present.

Summing up, we thus get for the total Hamiltonian

1° the energy of the system of nucleons and mesons without electromagnetic field, denoted, as in NF (56), by $\mathscr{H}_k + \mathscr{H}_F + \mathscr{H}_\Phi$,

2° the energy of the electromagnetic field $\mathscr{H}_f$ (formula (13) of Part I),

3° the total Coulomb energy of protons and charged mesons,

4° the interaction energy approximately given by

$$\left.\begin{aligned} &-\int \left(\vec{I}_{\text{nucl}} + \vec{I}_{\text{mes}}\right)\left(\vec{A}_{\perp} + \vec{A}_{\text{ex}}\right) dv \\ &+\int \left(\varrho_{\text{nucl}} + \varrho_{\text{mes}}\right) B_{\text{ex}}\, dv, \end{aligned}\right\} \tag{9}$$

the exact expression being obtained by adding to (9) the quadratic term (8).

Now in NF, Part II, § 5, arguments of principle have been developed for applying to the nucleon and meson variables a canonical transformation separating the static part of the meson fields. In fact, the situation in meson theory has been compared with that in electrodynamics and it has been pointed out that only the transformed variables provide a suitable starting point for a "correspondence" interpretation of the formalism. It further turns out that the transformation in question also brings about an appreciable simplification in the treatment of the electromagnetic properties of nuclear systems.

We have, thus, now to calculate the expression of the above Hamiltonian function in terms of the transformed variables. For the first part, this has been done in NF; the second part is unaffected by the canonical transformation; as regards the two last parts, we have simply to insert for the quantities $\varrho = \varrho_{\text{nucl}} + \varrho_{\text{mes}}$ and $\vec{I} = \vec{I}_{\text{nucl}} + \vec{I}_{\text{mes}}$ their expressions in terms of the new variables. In the following calculations, all symbols will be taken to denote the new variables and functions of the new variables; the old variables and the functions of those variables will then be distinguished by a $\overleftarrow{\ }$ (*e. g.* $\overleftarrow{A}$). Thus, calling $\mathcal{S}$ the canonical transformation operator, we write

$$\overleftarrow{A} = \mathcal{S}\, A\, \mathcal{S}^{-1}$$

and, according to NF (57, 20, 36),

$$\left.\begin{aligned} \mathcal{S} &= e^{\frac{i}{\hbar c}\mathcal{K}} \\ \mathcal{K} &= \int dv \left\{ \vec{\boldsymbol{F}}^{\circ}\, \vec{\boldsymbol{U}} - \vec{\boldsymbol{U}}^{\circ}\, \vec{\boldsymbol{F}} + \boldsymbol{\Psi}^{\circ}\, \boldsymbol{\Phi} \right\}; \end{aligned}\right\} \tag{10}$$

we may therefore write, with the notation introduced in NF (68),

$$\overleftarrow{A} = A + \sum_{l=1}^{\infty} \frac{1}{l!} \left\{ \frac{i}{\hbar c}\, \mathcal{K}, A \right\}^{l}. \tag{11}$$

It is just this formula we have to apply in order to express the quantities

$$\overleftarrow{\varrho} = \overleftarrow{\varrho}_{\text{nucl}} + \overleftarrow{\varrho}_{\text{mes}}, \quad \overleftarrow{\vec{I}} = \overleftarrow{\vec{I}}_{\text{nucl}} + \overleftarrow{\vec{I}}_{\text{mes}}, \tag{12}$$

occurring in the interaction terms of the Hamiltonian, in terms of the new variables. In the following sections, we shall treat the charge density and the current density separately, since the former has some peculiar properties not exhibited by the latter; we begin with the somewhat simpler case of the charge density.

## § 2. Expression of the charge density in terms of the transformed variables.

The first term in the expansion of type (11) of $\breve{\varrho}-\varrho$, *viz.* $\frac{i}{\hbar c}\left[\mathscr{K},\varrho\right]$, we shall denote by $\varrho_{(1)}$. In the first place, we have*, from (10) and (5),

$$\left.\begin{aligned}\varrho_{\times} &\equiv \frac{i}{\hbar c}\left[\mathscr{K},\varrho_{\text{mes}}\right]\\ &= \frac{e}{\hbar c}\left\{-\vec{\boldsymbol{F}}^{\circ}\wedge\vec{\boldsymbol{U}}+\vec{\boldsymbol{U}}^{\circ}\wedge\vec{\boldsymbol{F}}-\boldsymbol{\Psi}^{\circ}\wedge\boldsymbol{\Phi}\right\}_{\mathbf{3}}.\end{aligned}\right\}\tag{13}$$

The dependence of $\mathscr{K}$ on the variables pertaining to the different nucleons of the system may be expressed by

$$\mathscr{K}=\sum_i \boldsymbol{\tau}^{(i)}\int\mathfrak{F}^{(i)}\,dv;\tag{14}$$

comparing then the expressions (13) and (10) of $\varrho_{\times}$ and $\mathscr{K}$, we may also write

$$\varrho_{\times}=-\frac{e}{\hbar c}\sum_i\left\{\boldsymbol{\tau}^{(i)}\wedge\mathfrak{F}^{(i)}\right\}_{\mathbf{3}}\equiv\sum_i\varrho_{\times}^{(i)}.\tag{15}$$

Passing now on to $\varrho_{\text{nucl}}$, we get from (2)

$$\frac{i}{\hbar c}\left[\mathscr{K},\varrho_{\text{nucl}}\right]=-\frac{e}{2}\sum_i\delta\left(\vec{x}-\vec{x}^{(i)}\right)\cdot\frac{i}{\hbar c}\left[\mathscr{K},\tau_{\mathbf{3}}^{(i)}\right]$$

and from (14)

$$\frac{i}{\hbar c}\left[\mathscr{K},\tau_{\mathbf{3}}^{(i)}\right]=-\frac{2}{\hbar c}\left\{\boldsymbol{\tau}^{(i)}\wedge\int\mathfrak{F}^{(i)}\,dv\right\}_{\mathbf{3}},$$

whence, on account of (15),

$$\frac{i}{\hbar c}\left[\mathscr{K},\varrho_{\text{nucl}}\right]=-\sum_i\delta\left(\vec{x}-\vec{x}^{(i)}\right)\int\varrho_{\times}^{(i)}\,dv.\tag{16}$$

From (13) and (16), the term $\varrho_{(1)}$ is thus found to be

* We write $[A,B]$ for $[A,B]_{-}=AB-BA$.

$$\left.\begin{aligned} \varrho_{(1)} &= \varrho_{\times} - \sum_i \delta\left(\vec{x} - \vec{x}^{(i)}\right) \int \varrho_{\times}^{(i)}\, dv \\ &= \sum_i \left\{ \varrho_{\times}^{(i)} - \delta\left(\vec{x} - \vec{x}^{(i)}\right) \int \varrho_{\times}^{(i)}\left(\vec{x}'\right) dv' \right\}, \end{aligned}\right\} \tag{17}$$

giving by integration

$$\int \varrho_{(1)}\, dv = 0,$$

in agreement with the invariance property of the total charge of the system. The expression of $\varrho_{(1)}$ may be put into a more symmetrical shape by introducing the function of two sets of space variables

$$\varrho_{\times}\left(\vec{x}, \vec{x}'\right) \equiv \sum_i \varrho_{\times}^{(i)}\left(\vec{x}\right) \delta\left(\vec{x}' - \vec{x}^{(i)}\right); \tag{18}$$

we then get

$$\varrho_{(1)}\left(\vec{x}\right) = \int dv' \left[\varrho_{\times}\left(\vec{x}, \vec{x}'\right) - \varrho_{\times}\left(\vec{x}', \vec{x}\right)\right]. \tag{19}$$

Using (15) and the expressions NF (14) and NF (30) of the static meson fields, we may write for $\varrho_{\times}\left(\vec{x}, \vec{x}'\right)$

$$\left.\begin{aligned} &\varrho_{\times}\left(\vec{x}, \vec{x}'\right) \\ = -\frac{e}{\hbar c} &\left\{ \boldsymbol{N}\left(\vec{x}'\right) \wedge \vec{\boldsymbol{U}}\left(\vec{x}\right) - \boldsymbol{S}\left(\vec{x}'\right) \overset{\wedge}{\wedge} \vec{\boldsymbol{F}}\left(\vec{x}\right) + \vec{\boldsymbol{P}}\left(\vec{x}'\right) \wedge \boldsymbol{\Phi}\left(\vec{x}\right) \right\}_3 \\ &\cdot \operatorname{grad}' \varphi\left(\left|\vec{x}' - \vec{x}\right|\right). \end{aligned}\right\} \tag{20}$$

We see that $\varrho_{(1)}$ is linear in the free meson variables, so that it does not give any contribution to the charge density of a nuclear system in the absence of free mesons; it only gives rise to a peculiar interaction between nucleons and free mesons when an external electrostatic field (*e. g.* the field of atomic electrons) is present.

The next contribution to the expansion of $\overset{\frown}{\varrho} - \varrho$ is $\frac{i}{2\hbar c}\left[\mathscr{K}, \varrho_{(1)}\right]$. It consists of two terms of which the one,

$$\varrho_{(2)} = \sum_i \left\{ \varrho_{\times\times}^{(i)} - \delta\left(\vec{x} - \vec{x}^{(i)}\right) \int \varrho_{\times\times}^{(i)}\left(\vec{x}'\right) dv' \right\}, \tag{21}$$

is obtained by disregarding the non-commutability of the meson field variables and is thus quadratic in these variables, while the other*,

$$\left.\begin{aligned}\varrho_{\text{exch}} &= \frac{e}{2\hbar c}\left\{\vec{\boldsymbol{U}}^{\circ}\wedge\vec{\boldsymbol{F}}^{\circ}-\vec{\boldsymbol{F}}^{\circ}\wedge\vec{\boldsymbol{U}}^{\circ}\right\}_{3} = \frac{e}{\hbar c}\left\{\overline{\vec{\boldsymbol{U}}^{\circ}\wedge\vec{\boldsymbol{F}}^{\circ}}\right\}_{3} \\ &= -\frac{e}{2\hbar c}\operatorname{div}\left\{\vec{\boldsymbol{U}}^{\circ}\wedge\boldsymbol{V}^{\circ}-\boldsymbol{V}^{\circ}\wedge\vec{\boldsymbol{U}}^{\circ}\right\}_{3} = -\frac{e}{\hbar c}\operatorname{div}\left\{\overline{\vec{\boldsymbol{U}}^{\circ}\wedge\boldsymbol{V}^{\circ}}\right\}_{3},\end{aligned}\right\}\quad(22)$$

is independent of the meson field variables. The value (22) of $\varrho_{\text{exch}}$ is readily obtained from (13); in fact, the field-independent part of $\frac{i}{2\hbar c}[\mathscr{H}, \varrho_{\times}]$ is just the expression (22) and, since it may be written as a sum of terms relative to the single nucleons, and of which the space integrals vanish, it also represents, according to (15) and (17), the field-independent part of $\frac{i}{2\hbar c}[\mathscr{H}, \varrho_{(1)}]$. While, as indicated in formula (21), the quadratic term $\varrho_{(2)}$ is obviously a sum of contributions from the single nucleons, the field-independent term is, according to (22), of the form $\sum_{i \neq k}\left\{\boldsymbol{\tau}^{(i)}\wedge\boldsymbol{\tau}^{(k)}\right\}_{3}\Phi^{(i,k)}$ and does, therefore, not contain any such contribution, but presents an "exchange" character with regard to the proton and neutron states of the nucleons; we shall call it the "exchange charge density". For a pair of nucleons, exchange operators of the type just considered, $\left\{\boldsymbol{\tau}^{(1)}\wedge\boldsymbol{\tau}^{(2)}\right\}_{3}\Phi^{(1,2)}$, have non-vanishing matrix elements only if the system is a deuteron, and these matrix elements refer to states in which the two particles have exchanged their proton or neutron character; in particular, the mean value of such an operator in any stationary state of the system is zero.

For the $\varrho^{(i)}_{\times\times}$ entering into the expression (21) of the quadratic term $\varrho_{(2)}$, we may write, according to (14) and (15),

$$\varrho^{(i)}_{\times\times}(\vec{x}) = -\frac{e}{\hbar c}\cdot\frac{i}{2\hbar c}\int dv'\left[\boldsymbol{\tau}^{(i)}\mathfrak{J}^{(i)}(\vec{x}');\ \left\{\boldsymbol{\tau}^{(i)}\wedge\mathfrak{J}^{(i)}(\vec{x})\right\}_{3}\right]_{-}.$$

The commutator in this formula has the following value (the index $(i)$ having been dropped):

* We denote by $\overline{AB}$ the symmetrical combination $\frac{1}{2}(AB + BA)$.

$$\left[\mathcal{T}_1', \mathcal{T}_2\right]_- - \left[\mathcal{T}_2', \mathcal{T}_1\right]_- + i\tau_1\left[\mathcal{T}_3', \mathcal{T}_1\right]_+ + i\tau_2\left[\mathcal{T}_3', \mathcal{T}_2\right]_+ \\ - i\tau_3\left\{\left[\mathcal{T}_1', \mathcal{T}_1\right]_+ + \left[\mathcal{T}_2', \mathcal{T}_2\right]_+\right\};$$

putting

$$\left\{\begin{array}{l} \mathfrak{J}^{(i)} \equiv \boldsymbol{a}^{(i)} + \vec{\sigma}^{(i)} \vec{\boldsymbol{b}}^{(i)} \\ \boldsymbol{a}^{(i)} \equiv g_1 \vec{f}^{(i)} \vec{\boldsymbol{U}} \\ \vec{\boldsymbol{b}}^{(i)} \equiv \frac{g_2}{\varkappa} \varrho_3^{(i)} \vec{f}^{(i)} \wedge \vec{\boldsymbol{F}} + \frac{f_2}{\varkappa} \vec{f}^{(i)} \boldsymbol{\Phi} \\ \vec{f}^{(i)} \equiv \operatorname{grad}^{(i)} \varphi\left(\left|\vec{x} - \vec{x}^{(i)}\right|\right), \end{array}\right. \tag{23}$$

we get, $\boldsymbol{a}^{(i)}$ and $\vec{\boldsymbol{b}}^{(i)}$ being commutable,

$$\left\{\begin{array}{l} \left[\mathcal{T}_i', \mathcal{T}_j\right]_- = 2 i \vec{\sigma} \vec{b}_i' \wedge \vec{b}_j \\ \left[\mathcal{T}_i', \mathcal{T}_j\right]_+ = 2\left[a_i' a_j + \vec{b}_i' \vec{b}_j + \vec{\sigma}\left(a_i' \vec{b}_j + \vec{b}_i' a_j\right)\right], \end{array}\right.$$

so that finally

$$\left.\begin{array}{l} \varrho_{\times\times}^{(i)} = \frac{e}{(\hbar c)^2} \int dv' \left\{\vec{\sigma}^{(i)} \left(\vec{\boldsymbol{b}}^{(i)'} \wedge \vec{\boldsymbol{b}}^{(i)}\right)_3 \right. \\ \qquad + \boldsymbol{\tau}^{(i)}\left[a_3^{(i)'} \boldsymbol{a}^{(i)} + \vec{b}_3^{(i)'} \vec{\boldsymbol{b}}^{(i)} + \vec{\sigma}^{(i)}\left(a_3^{(i)'} \vec{\boldsymbol{b}}^{(i)} + \vec{b}_3^{(i)'} \boldsymbol{a}^{(i)}\right)\right] \\ \qquad \left. - \boldsymbol{\tau}_3^{(i)}\left[\boldsymbol{a}^{(i)'} \boldsymbol{a}^{(i)} + \vec{\boldsymbol{b}}^{(i)'} \vec{\boldsymbol{b}}^{(i)} + \vec{\sigma}^{(i)}\left(\boldsymbol{a}^{(i)'} \vec{\boldsymbol{b}}^{(i)} + \vec{\boldsymbol{b}}^{(i)'} \boldsymbol{a}^{(i)}\right)\right]\right\}. \end{array}\right\} \tag{24}$$

Owing to the existence of the fluctuating zero-point meson field, this expression gives a contribution to the charge density operator of a nuclear system in the absence of free mesons, namely the mean value of $\varrho_{(2)}$ with respect to the meson variables for the state corresponding to the zero-point meson field. Denoting such a mean value by $\mathfrak{M}_0\{\cdots\cdot\}$ and taking account of the formulae (A 7) and (A 8) of the Appendix, we directly get from (24)

$$\begin{aligned} \mathfrak{M}_0\left\{\varrho_{\times\times}^{(i)}\right\} &= -\frac{2e}{(\hbar c)^2} \tau_3^{(i)} \int dv' \, \mathfrak{M}_0\left\{a_1^{(i)'} a_1^{(i)} + \vec{b}_1^{(i)'} \vec{b}_1^{(i)}\right\} \\ &= -\frac{2e}{(\hbar c)^2} \tau_3^{(i)} \int dv' \left[g_1^2 \, \mathfrak{M}_0\left\{\left(\vec{f}^{(i)} \vec{U}_1\right)_{\vec{x}} \left(\vec{f}^{(i)} \vec{U}_1\right)_{\vec{x}'}\right\}\right. \\ &\quad + \left(\frac{g_2}{\varkappa}\right)^2 \mathfrak{M}_0\left\{\left(\vec{f}^{(i)} \wedge \vec{F}_1\right)_{\vec{x}} \left(\vec{f}^{(i)} \wedge \vec{F}_1\right)_{\vec{x}'}\right\} \\ &\quad \left. + \left(\frac{f_2}{\varkappa}\right)^2 \left(\vec{f}^{(i)} \vec{f}^{(i)'}\right) \mathfrak{M}_0\left\{\Phi_1 \Phi_1'\right\}\right], \end{aligned}$$

according to (23). By means of the formulae (A 12), (A 13), and (A 14) of the Appendix, this takes the form

$$\left.\begin{aligned} &\mathfrak{M}_0\left\{\varrho^{(i)}_{\times\times}\right\} \\ &= -e\,\tau_3^{(i)}\cdot\frac{g_1^2+2g_2^2+f_2^2}{4\pi\hbar c}\cdot\frac{e^{-\kappa r^{(i)}}}{r^{(i)}}\left(\kappa+\frac{1}{r^{(i)}}\right)\frac{1}{2\pi^2 r^{(i)}}K_2\left(\kappa r^{(i)}\right), \end{aligned}\right\} \quad (25)$$

where $r^{(i)} = \left|\vec{x}-\vec{x}^{(i)}\right|$, and $K_2$ is the Bessel function defined in the Appendix (of course, we have in our theory the relation $f_2^2 = g_2^2$, but we have not made use of it in writing this formula in order to distinguish the contribution from pseudoscalar mesons). Owing to the singularity of $K_2(z)$ for $z = 0$, however, the integral $\int dv\,\mathfrak{M}_0\left\{\varrho^{(i)}_{\times\times}\right\}$ diverges, so that $\mathfrak{M}_0\left\{\varrho_{(2)}\right\}$ is not well-defined. According to the general criterium proposed in NF, Part II, § 5, no significance can therefore be attributed to any effect depending on this term or similar effects arising from a perturbation calculation, and it is *a fortiori* useless to consider further terms in the expansion of $\overset{\smile}{\varrho}-\varrho$.

In particular, one might think that an additional interaction energy between a nucleon and an electron would be obtained by taking for $B_{ex}$ in $\int dv\,B_{ex}\,\mathfrak{M}_0\left\{\varrho_{(2)}\right\}$ the Coulomb potential

$$B_{ex} = -\frac{e}{4\pi\left|\vec{x}-\vec{x}^{(0)}\right|}$$

of an electron at $\vec{x}^{(0)}$, since it is possible in this case to carry out the integration in such a way as to get a finite result. In fact, taking the position of the nucleon as the origin of polar coordinates and observing that $\mathfrak{M}_0\left\{\varrho^{(i)}_{\times\times}\right\}$ is a spherically symmetrical charge distribution, we may write, according to potential theory,

$$\begin{aligned} &\int\frac{1}{4\pi\left|\vec{x}-\vec{x}^{(0)}\right|}\,\mathfrak{M}_0\left\{\varrho^{(i)}_{\times\times}\right\}r^2\,dr\,d\Omega \\ &= \frac{1}{R}\int_0^R\mathfrak{M}_0\left\{\varrho^{(i)}_{\times\times}\right\}r^2\,dr+\int_R^\infty\frac{1}{r}\,\mathfrak{M}_0\left\{\varrho^{(i)}_{\times\times}\right\}r^2\,dr, \end{aligned}$$

where $R \equiv \left|\vec{x}^{(i)}-\vec{x}^{(0)}\right|$; therefore

$$\int B_{\text{ex}}\, \mathfrak{M}_0 \left\{\varrho_{(2)}\right\} dv = -e \int_R^\infty \left(\frac{1}{r} - \frac{1}{R}\right) \mathfrak{M}_0 \left\{\varrho^{(i)}_{\times\times}\right\} r^2\, dr.$$

To this should then be added a term of the same order of magnitude, derived as a second order perturbation from the other couplings between nucleons and mesons; we need not consider it here, however, but refer the reader to the forthcoming paper by J. Serpe mentioned in the Introduction. The total effect has been first pointed out by H. Fröhlich, W. Heitler and B. Kahn [16], but its reality was contested by W. Lamb [17]. From our point of view, it should be clear that the whole effect must be discarded; a more detailed discussion will be found in Serpe's paper.

The conclusions of this section may thus be summarized in the formula

$$\overleftarrow{\varrho} = \varrho_{\text{nucl}} + \varrho_{\text{mes}} + \varrho_{(1)} + \varrho_{(2)} + \varrho_{\text{exch}}, \tag{26}$$

the various terms being defined by (2), (5), (17) [with (13), (15)], or (19) [with (20)], (21) [with (24), (23)], and (22), respectively.

## § 3. Electric dipole and quadrupole moments.

With the help of (26), the expressions of the electric dipole and quadrupole moments, defined by formulae (32) of Part I, may be written down immediately in the same form. They comprise 1° terms independent of the meson fields (which we often call, for the sake of brevity, "field-independent terms"), *viz.* a term due to the elementary charges of the protons ($\varrho_{\text{nucl}}$) and an exchange term arising from the coupling through the meson field (as embodied in $\varrho_{\text{exch}}$); 2° a coupling term between nucleons and free mesons, linear in the meson field components (from $\varrho_{(1)}$); 3° terms quadratic in the meson variables, *viz.* the contribution from the free mesons ($\varrho_{\text{mes}}$) and a quadratic coupling term (from $\varrho_{(2)}$).

It will be noticed that the spreading of the charge of a nucleon, which would be due to the interaction with the zero-point meson field, a part of which would be given by $\mathfrak{M}_0\left\{\varrho_{(2)}\right\}$, would possess spherical symmetry and would thus not give rise to any multipole moments (as usually defined; cf. footnote on p. 21);

in fact, it would act only at large distances from the nucleon and the analysis of Part I, § 4 would not be applicable to it*.

The most notable terms characteristic of meson theory are the exchange terms which, using (22), NF (14, 3, 4, 89), and the relation

$$\int \varphi\left(\left|\vec{x}^{(i)}-\vec{x}\right|\right) \varphi\left(\left|\vec{x}^{(k)}-\vec{x}\right|\right) \vec{x}\, dv = \frac{e^{-\varkappa\left|\vec{x}^{(i)}-\vec{x}^{(k)}\right|}}{8\pi\varkappa} \cdot \frac{\vec{x}^{(i)}+\vec{x}^{(k)}}{2},$$

may be written

$$\left.\begin{aligned} \vec{P}_{\mathrm{exch}} = \frac{e}{\hbar c}\int \overline{\left\{\vec{\boldsymbol{U}}^{\circ} \wedge \boldsymbol{V}^{\circ}\right\}_3}\, dv = \frac{1}{2}\cdot\frac{e}{\hbar c}\cdot\frac{g_1 g_2}{\varkappa}\sum_{i \neq k}\left\{\boldsymbol{\tau}^{(i)} \wedge \boldsymbol{\tau}^{(k)}\right\}_3 \\ \left[\vec{\sigma}^{(i)} \wedge \left(\vec{x}^{(i)}-\vec{x}^{(k)}\right)\right] \varphi\left(\left|\vec{x}^{(i)}-\vec{x}^{(k)}\right|\right) \end{aligned}\right\} \quad (27)$$

$$\left.\begin{aligned} Q^{lm}_{\mathrm{exch}} &= \frac{1}{2}\frac{e}{\hbar c}\int\left[\overline{\left\{\vec{\boldsymbol{U}}^{\circ} \wedge \boldsymbol{V}^{\circ}\right\}^{l}_3} x^m + \overline{\left\{\vec{\boldsymbol{U}}^{\circ} \wedge \boldsymbol{V}^{\circ}\right\}^{m}_3} x^l\right] dv \\ &= \frac{1}{8}\cdot\frac{e}{\hbar c}\cdot\frac{g_1 g_2}{\varkappa}\sum_{i \neq k}\left\{\boldsymbol{\tau}^{(i)} \wedge \boldsymbol{\tau}^{(k)}\right\}_3 \cdot \left[\left(\vec{x}^{(i)}+\vec{x}^{(k)}\right)^l \left(\vec{\sigma}^{(i)} \wedge \left(\vec{x}^{(i)}-\vec{x}^{(k)}\right)\right)^m \right. \\ &\quad \left. + \left(\vec{x}^{(i)}+\vec{x}^{(k)}\right)^m \left(\vec{\sigma}^{(i)} \wedge \left(\vec{x}^{(i)}-\vec{x}^{(k)}\right)\right)^l\right] \cdot \varphi\left(\left|\vec{x}^{(i)}-\vec{x}^{(k)}\right|\right). \end{aligned}\right\} \quad (28)$$

Owing to their exchange character, these terms do not give any contribution to the energy of stationary states of nuclear systems, but they may play a role in the calculation of the transition probabilities between such states under the influence of an external electric field. Since the mean value of the charge density $\varrho_{\mathrm{nucl}}$ in any stationary state is invariant with respect to reflections about the centre of the nucleus, there is no electric dipole moment at all in these states. The interaction energy with an external electric field is thus determined, apart from the total charge, by the quadrupole moment $Q_{\mathrm{nucl}}$. A most important example of such an interaction is met with in the theory of hyperfine structure, the external field in that case being the field of the atomic electrons. It may be shown** that the

* The effect of a (finite) spreading of the nuclear charge on the atomic levels (owing to the electrostatic interaction with the atomic electrons) is discussed by H. Casimir, loc. cit. [18], § 5.

** See H. Casimir, loc. cit. [18], § 4. The expression (*d*) which forms the starting point of Casimir's investigation is identical with our interaction term $-(Q\,\mathrm{grad}_0)\vec{\mathfrak{E}}_0$, where $\vec{\mathfrak{E}}_0$ is the field due to the atomic electrons at the centre of the nucleus.

quadrupole charge distribution of the nucleus enters into the expression of the interaction energy only through one parameter, called the "quadrupole moment" of the nucleus (in the spectroscopic sense), and conveniently defined as 6 times the mean value of our $Q^{33} - \frac{1}{3}\sum_i Q^{ii}$ for the state with the maximum value of the magnetic quantum number*.

For the calculation of transition probabilities between states of the same total energy, it is equivalent, as noticed at the end of § 4 of Part I, as operator of electric dipole interaction to use $-\vec{A}_0 \dot{\vec{P}}$ instead of $-\vec{E}_0 \vec{P}$. At an early stage of the development of nuclear theory, it was pointed out by SIEGERT [19] that the choice of the former operator might be advantageous because of a remarkable connexion between the operator** $\dot{\vec{P}}$ $\left(\text{or} \int \vec{I}\, dv\right)$ and the exchange potential of the nuclear forces; the exchange part of $\dot{\vec{P}}$ could then directly be written down when this potential was given. We shall now derive "SIEGERT's theorem" from the more general point of view of our theory which also involves "velocity-dependent" couplings between the nucleons (*i. e.* terms depending on the velocities of the nucleons). For this purpose (since SIEGERT's theorem is only concerned with processes involving no free mesons), we have only to consider the field-independent part of $\dot{\vec{P}}$; neglecting accordingly all terms in $\vec{P}$ which are quadratic in the meson field variables, we have only to calculate (cf. (26)) the field-independent part of $\dot{\vec{P}}_{\text{nucl}} + \dot{\vec{P}}_{(1)} + \dot{\vec{P}}_{\text{exch}}$. We begin with $\dot{\vec{P}}_{\text{nucl}}$; from

$$\vec{P}_{\text{nucl}} = e \sum_i \frac{1-\tau_3^{(i)}}{2} \vec{x}^{(i)}$$

we get

* This quantity has been calculated for the ground state of the deuteron in NF, Part III, § 3 (no restriction being made on the nuclear source constants; cf. [4]). It should be observed, however, that, since relative coordinates were used, the quantity $Q$ there computed (formula (123)) should be divided by 4 to yield the quadrupole moment in the spectroscopic sense.

** In conformity with our previous notations, $\vec{I}$ represents the total current density of the system, including the dependence on the vector-potential.

$$\dot{\vec{P}}_{\text{nucl}} = e \sum_i \frac{1-\tau_3^{(i)}}{2} \vec{\alpha}^{(i)} - \frac{e}{2} \sum_i \dot{\tau}_3^{(i)} \vec{x}^{(i)},$$

the field-independent part of which to a sufficient approximation reduces to

$$\mathfrak{M}_0 \left\{ \dot{\vec{P}}_{\text{nucl}} \right\} = \int \vec{I}_{\text{nucl}}\, dv - \frac{e}{\hbar c} \sum_i \frac{i}{2} \left[ \mathcal{V}_n + \mathcal{W}_n,\ \tau_3^{(i)} \right] \vec{x}^{(i)}. \quad (29)$$

In this formula, $\mathcal{V}_n$ is the static potential of the nuclear forces given by NF (65), and $\mathcal{W}_n$ the velocity-dependent potential given by NF (84)* or NF (85):

$$\left.\begin{aligned} \mathcal{V}_n &= \frac{1}{2} \int (\boldsymbol{N}' \boldsymbol{N} + \kappa^2 \boldsymbol{S}' \boldsymbol{S})\, \varphi\, dv\, dv' \\ \mathcal{W}_n &= \int \left( \boldsymbol{N}' \vec{\boldsymbol{T}} + \vec{\boldsymbol{S}}' \wedge \vec{\boldsymbol{M}} + \vec{\boldsymbol{P}}' \boldsymbol{R} \right) \operatorname{grad} \varphi\, dv\, dv'. \end{aligned}\right\} \quad (30)$$

All their terms are of the form $\int \boldsymbol{A}\left(\vec{x}'\right) \boldsymbol{B}\left(\vec{x}\right) dv\, dv'$ with

$$\boldsymbol{A}\left(\vec{x}\right) = \sum_i \boldsymbol{\tau}^{(i)} a^{(i)} \delta\left(\vec{x} - \vec{x}^{(i)}\right)$$

and an analogous expression for $\boldsymbol{B}$. We have

$$\sum_i \frac{i}{2} \left[ \boldsymbol{A}\left(\vec{x}'\right) \boldsymbol{B}\left(\vec{x}\right),\ \tau_3^{(i)} \right] \vec{x}^{(i)} = \left\{ \boldsymbol{A}\left(\vec{x}'\right) \wedge \boldsymbol{B}\left(\vec{x}\right) \right\}_3 \left(\vec{x} - \vec{x}'\right)$$

and, therefore,

$$\left.\begin{aligned} \vec{J} &= -\frac{e}{\hbar c} \sum_i \frac{i}{2} \left[ \mathcal{V}_n,\ \tau_3^{(i)} \right] \vec{x}^{(i)} \\ &= -\frac{e}{2\hbar c} \int \left\{ \boldsymbol{N}' \wedge \boldsymbol{N} + \kappa^2 \vec{\boldsymbol{S}}' \wedge \vec{\boldsymbol{S}} \right\}_3 \left(\vec{x} - \vec{x}'\right) \varphi\, dv\, dv', \end{aligned}\right\} \quad (31)$$

$$\left.\begin{aligned} \vec{J}' &= -\frac{e}{\hbar c} \sum_i \frac{i}{2} \left[ \mathcal{W}_n,\ \tau_3^{(i)} \right] \vec{x}^{(i)} \\ &= -\frac{e}{\hbar c} \int \left( \left\{ \boldsymbol{N}' \wedge \vec{\boldsymbol{T}} + \vec{\boldsymbol{S}}' \mathop{\wedge}^{\wedge} \vec{\boldsymbol{M}} + \vec{\boldsymbol{P}}' \wedge \boldsymbol{R} \right\}_3 \cdot \operatorname{grad} \varphi \right) \left(\vec{x} - \vec{x}'\right) dv\, dv', \end{aligned}\right\} \quad (32)$$

$$\mathfrak{M}_0 \left\{ \dot{\vec{P}}_{\text{nucl}} \right\} = \int \vec{I}_{\text{nucl}}\, dv + \vec{J} + \vec{J}'. \quad (33)$$

Looking apart from the relatively small term $\vec{J}'$, formula (33) already embodies the primitive form of Siegert's theorem;

* The second formula (30) is equivalent with NF (84), though slightly modified in form.

the comparison of (31) and (30) shows how the exchange part of $\mathfrak{M}_0 \left\{ \dot{\vec{P}}_{\text{nucl}} \right\}$ $\left( \text{or of } \mathfrak{M}_0 \left\{ \int \dot{\vec{I}}\, dv \right\} \right)$ may be built up from the static potential $\mathcal{V}_n$. We may, however, obtain a more rigorous formulation by taking account of the other terms of $\mathfrak{M}_0 \left\{ \dot{\vec{P}} \right\}$, namely $\mathfrak{M}_0 \left\{ \dot{\vec{P}}_{(1)} + \dot{\vec{P}}_{\text{exch}} \right\}$. We have from (19)

$$\vec{P}_{(1)} = \int dv\, dv'\, \varrho_{\times} \left( \vec{x}, \vec{x}' \right) \left( \vec{x} - \vec{x}' \right),$$

whence, by (20),

$$\left. \begin{aligned} &\mathfrak{M}_0 \left\{ \dot{\vec{P}}_{(1)} \right\} \\ &= -\frac{e}{\hbar c} \int dv\, dv'\, \mathfrak{M}_0 \left\{ \left( \boldsymbol{N}' \wedge \dot{\vec{\boldsymbol{U}}} - \vec{\boldsymbol{S}}' \overset{\wedge}{\wedge} \dot{\vec{\boldsymbol{F}}} + \vec{\boldsymbol{P}}' \wedge \dot{\boldsymbol{\Phi}} \right)_{\mathbf{3}} \text{grad}'\, \varphi \right\} \left( \vec{x} - \vec{x}' \right). \end{aligned} \right\} \quad (34)$$

The contribution from $\dot{\vec{P}}_{\text{exch}}$ may be put into a quite analogous form by writing, from (22) and NF (9, 14),

$$\begin{aligned} \vec{P}_{\text{exch}} &= \frac{e}{\hbar c} \int dv \overline{\left\{ \vec{\boldsymbol{U}}^{\circ} \wedge \vec{\boldsymbol{F}}^{\circ} \right\}}_{\mathbf{3}}\, \vec{x} \\ &= \frac{e}{2\hbar c} \int dv\, dv' \left( \left\{ \vec{\boldsymbol{S}}' \overset{\wedge}{\wedge} \vec{\boldsymbol{F}}^{\circ} - \boldsymbol{N}' \wedge \vec{\boldsymbol{U}}^{\circ} \right\}_{\mathbf{3}} \cdot \text{grad}'\, \varphi \right) \left( \vec{x} - \vec{x}' \right), \end{aligned}$$

the replacement of the factor $\vec{x}$ by $\left( \vec{x} - \vec{x}' \right)$ in the last formula being allowed on account of rot $\vec{\boldsymbol{F}}^{\circ} = 0$, div $\vec{\boldsymbol{U}}^{\circ} = 0$. Thence,

$$\mathfrak{M}_0 \left\{ \dot{\vec{P}}_{\text{exch}} \right\} = -\frac{e}{\hbar c} \int dv\, dv' \left( \left\{ \boldsymbol{N}' \wedge \overset{\odot}{\vec{\boldsymbol{U}}}{}^{\circ} - \boldsymbol{S}' \overset{\wedge}{\wedge} \overset{\odot}{\vec{\boldsymbol{F}}}{}^{\circ} \right\}_{\mathbf{3}} \cdot \text{grad}'\, \varphi \right) \left( \vec{x} - \vec{x}' \right), \quad (35)$$

with the notation (slightly extending NF (79))

$$\overset{\odot}{A} = \frac{i}{\hbar c} \left[ \mathcal{H}_n + \mathcal{W}_n, A \right].$$

Now, the meson field equations* yield to our approximation,

$$\left. \begin{aligned} \mathfrak{M}_0 \left\{ \dot{\vec{\boldsymbol{U}}} \right\} + \overset{\odot}{\vec{\boldsymbol{U}}}{}^{\circ} &= \vec{\boldsymbol{T}} \\ \mathfrak{M}_0 \left\{ \dot{\vec{\boldsymbol{F}}} \right\} + \overset{\odot}{\vec{\boldsymbol{F}}}{}^{\circ} &= -\vec{\boldsymbol{M}} \\ \mathfrak{M}_0 \left\{ \dot{\boldsymbol{\Phi}} \right\} \qquad &= \boldsymbol{R}. \end{aligned} \right\} \quad (36)$$

* Cf. the equations NF (87) which must be completed, however, by the contributions from the term $\mathcal{W}$ in NF (86), $\mathcal{W}$ being given by NF (67).

Inserting the values (36) in the sum of (34) and (35), and comparing with (32), we get just

$$\mathfrak{M}_0 \left\{ \dot{\vec{P}}_{(1)} + \dot{\vec{P}}_{\mathrm{exch}} \right\} = - \vec{J}'. \tag{37}$$

Addition of (33) and (37) then gives the generalized form of SIEGERT's theorem

$$\mathfrak{M}_0 \left\{ \dot{\vec{P}} \right\} = \int \vec{I}_{\mathrm{nucl}}\, dv + \vec{J}. \tag{38}$$

It is remarkable that the exchange part of $\mathfrak{M}_0 \left\{ \dot{\vec{P}} \right\}$ is completely determined by the static potential, even when due account is taken of the effects of the first order in the velocities of the nucleons. From the primitive form (33) of the theorem, with the term $\vec{J}'$ omitted, it had been concluded [20] that it was justified to take as electric dipole interaction the operator $-\vec{E}_0 \vec{P}_{\mathrm{nucl}}$ without any exchange moment. Although this argumentation is not rigorous, we see that the conclusion is nevertheless correct even to the first order of approximation with respect to the nuclear velocities. From the preceding analysis it appears, more precisely, that (since the relevant matrix elements of $-\vec{E}_0 \vec{P}_{(1)}$ or $-\vec{A}_0 \dot{\vec{P}}_{(1)}$ vanish) the contribution from $-\vec{E}_0 \vec{P}_{\mathrm{exch}}$ can only be of a higher order in the velocities of the nucleons*—at first sight a somewhat surprising result, since the operator itself is velocity-independent—and that this contribution just cancels the term $\vec{J}'$ of the same order of magnitude from $-\vec{E}_0 \vec{P}_{\mathrm{nucl}}$. Summarizing, we may say that the transition probabilities between states of the same energy, insofar as electric dipole interaction is concerned, may, to the first order in nuclear velocities, indifferently be computed from the operator $-\vec{E}_0 \vec{P}_{\mathrm{nucl}}$ or from the operator $-\vec{A}_0 \mathfrak{M}_0 \left\{ \dot{\vec{P}} \right\}$, given by (38) and (31). In the next section, we shall verify by a direct calculation that, as results from the way it has been introduced, $\vec{J}$ is just the "exchange" part of the integral current operator.

* In connexion with the problem of the photo-disintegration of the deuteron, treated from the point of view of the present paper, this point has been discussed in detail by A. PAIS, loc. cit. [7], Appendix.

We may finally remark that SIEGERT's theorem also holds in any meson theory using only one kind of meson field, such as a pure vector or a pure pseudoscalar theory. In these cases, the expression of $\vec{J}$ would involve a term arising from the dipole interaction; in the mixed theory, such terms automatically cancel each other, yielding a simpler form for $\vec{J}$ (cf. the Appendix). It may therefore be said that the simplification of the electrical quantities in the mixed theory has its origin in SIEGERT's theorem on account of the simpler determination of the static potential.

## § 4. The current density.

The expression of the current density in terms of the new variables may again be obtained as an expansion of the type (11). We shall not here carry out a complete calculation, but confine ourselves to the terms which are of practical interest; thus, we shall, in this expansion, neglect all terms which, besides involving some power of the source constants $g$ and $f$ also contain a factor of the order of magnitude of the ratio of the nucleon velocities to the velocity of light. This implies, in the first place, that in all terms of the form

$$\left\{\frac{i}{\hbar c}\,\mathscr{K}, \vec{I}_{\text{nucl}} + \vec{I}_{\text{mes}}\right\}^{l}$$

we shall neglect the contributions arising from $\vec{I}_{\text{nucl}}$. We may then write to a sufficient approximation

$$\left.\begin{aligned} \vec{I} &= \vec{I}_{\text{nucl}} + \vec{I}_{\text{mes}} + \vec{I}_{\times} + \frac{i}{2\hbar c}\left[\mathscr{K}, \vec{I}_{\times}\right], \\ \text{with}\quad \vec{I}_{\times} &= \frac{i}{\hbar c}\left[\mathscr{K}, \vec{I}_{\text{mes}}\right]. \end{aligned}\right\} \tag{39}$$

As regards the dependence on the meson variables, the situation is here slightly complicated by the fact that $\vec{I}_{\text{mes}}$ itself, apart from the current density for a system of entirely free mesons,

$$\vec{I}_{\text{free mes}} = \frac{e}{\hbar c}\left\{\operatorname{rot}\vec{\boldsymbol{U}} \wedge \vec{\boldsymbol{U}} + \kappa^{-2}\vec{\boldsymbol{F}} \wedge \operatorname{div}\vec{\boldsymbol{F}} - \operatorname{grad}\boldsymbol{\Psi} \wedge \boldsymbol{\Psi}\right\}_3, \tag{40}$$

also contains a term, linear in the meson variables, which depends on the nuclear source densities:

$$\vec{I}_{\text{mes}} = \vec{I}_{\text{free mes}} + \frac{e}{\hbar c}\left\{\vec{\boldsymbol{S}} \overset{\wedge}{\wedge} \vec{\boldsymbol{U}} - \kappa^{-2} \vec{\boldsymbol{F}} \wedge \boldsymbol{N} + \vec{\boldsymbol{P}} \wedge \boldsymbol{\Psi}\right\}_{\mathbf{3}}. \tag{41}$$

Therefore, $\vec{I}_{\times}$ as well as $\frac{i}{2\hbar c}\left[\mathscr{H}, \vec{I}_{\times}\right]$ will contain terms of degrees 0,1 and 2 in the meson variables; for those of degree zero* we readily get, from (40) and (41),

$$\vec{I}_{(0)} = \frac{e}{\hbar c}\left\{\overline{\vec{\boldsymbol{G}}^{\circ} \overset{\wedge}{\wedge} \vec{\boldsymbol{U}}^{\circ}} - \overline{\vec{\boldsymbol{F}}^{\circ} \wedge \boldsymbol{V}^{\circ}} + \overline{\vec{\boldsymbol{\Gamma}}^{\circ} \wedge \boldsymbol{\Psi}^{\circ}}\right\}_{\mathbf{3}}. \tag{42}$$

We may thus write, instead of (39),

$$\vec{I} = \vec{I}_{\text{nucl}} + \vec{I}_{\text{free mes}} + \vec{I}_{(1)} + \vec{I}_{(2)} + \vec{I}_{(0)}, \tag{43}$$

$\vec{I}_{(1)}$ and $\vec{I}_{(2)}$ denoting expressions linear and quadratic, respectively, in the meson variables, which need not be written down explicitly.

The field-independent term $\vec{I}_{(0)}$ is not entirely analogous to the term $\varrho_{\text{exch}}$ of formula (26), for the expression on the right in formula (42) gives rise not only to an "exchange" current density $\vec{I}_{\text{exch}}$ (involving a sum over all pairs of nucleons), but also to a "proper" current density, $\vec{I}_{\text{prop}}$, consisting of a sum of terms pertaining to the single nucleons. Since we have assumed for the nucleons a theory corresponding to the idealization of material points, the proper current density $\vec{I}_{\text{prop}}$ will contain divergent contributions. Such divergences, however, have a quite different origin from those arising from the field-independent part $\mathfrak{M}_0\left\{\vec{I}_{(2)}\right\}$ of $\vec{I}_{(2)}$ on account of the existence of the fluctuating zero-field. While, according to our general criterium, we must entirely discard the latter, we may therefore treat the former quite independently and, for instance, prevent their occurrence

* It should be noticed that the separation of the terms of degrees 0 and 2 is not unambiguous, since a commutation of the variables $\vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{U}}$ in the latter would, on account of the commutation rules, give rise to a contribution of order zero. An unambiguous definition is first obtained if one adds the condition that the terms in question be Hermitian. Such an Hermitization has been performed in formula (42), as indicated by the symmetrization bars.

by replacing the point distribution of the nuclear sources by some extended distribution. To this question we shall come back in the next section, in connexion with the problem of the proper magnetic moment of nucleons. Here, we shall only point out that the expression of the integral field-independent current $\int \vec{I}_{(0)}\, dv$ turns out to be identical with the quantity $\vec{J}$ defined by (31); the details of the calculation are reported in the Appendix. Owing to the factor $(\vec{x}-\vec{x}')$ in $\vec{J}$, no "proper" contribution obviously arises in this case, so that we actually have

$$\int \vec{I}_{(0)}\, dv = \int \vec{I}_{\mathrm{exch}}\, dv = \vec{J}. \tag{44}$$

## § 5. Magnetic dipole moment.

The last point we have to discuss is the transformation of the magnetic dipole moment of a system of mesons and nucleons to the new variables. For the sake of completeness, we begin by briefly recalling the well-known situation as regards Dirac particles and free mesons. Consider a Dirac particle of charge $e$ and mass $M$; let

$$\vec{v}^{(0)} = \frac{1}{Mc}\left(\vec{p}^{(0)} - e\,\vec{A}\left(\vec{x}^{(0)}\right)\right),$$

$\vec{x}^{(0)}$, $\vec{p}^{(0)}$ being the canonical position and momentum coordinates of the particle (the latter multiplied by $c$). The evaluation of

$$\frac{d}{c\,dt}\left\{\varrho_2\,\vec{\sigma}\,\delta\left(\vec{x}-\vec{x}^{(0)}\right)\right\} = \frac{1}{i\hbar c}\left[\varrho_2\,\vec{\sigma}\,\delta\left(\vec{x}-\vec{x}^{(0)}\right),\, \mathscr{H}_k\right]$$

with the help of the Hamiltonian

$$\mathscr{H}_k = Mc\left(\vec{\alpha}\,\vec{v}^{(0)} + \varrho_3\, c\right)$$

leads, after some easy reductions, to the formula

$$\vec{I} = e\,\vec{\alpha}\,\delta\left(\vec{x}-\vec{x}^{(0)}\right) = \frac{e}{c}\,\varrho_3\cdot\frac{1}{2}\left(\vec{v}^{(0)}\,\delta\left(\vec{x}-\vec{x}^{(0)}\right) + \delta\left(\vec{x}-\vec{x}^{(0)}\right)\vec{v}^{(0)}\right)$$
$$+\,\mu_0\,\mathrm{rot}\left\{\varrho_3\,\vec{\sigma}\,\delta\left(\vec{x}-\vec{x}^{(0)}\right)\right\} + \mu_0\,\frac{d}{c\,dt}\left\{\varrho_2\,\vec{\sigma}\,\delta\left(\vec{x}-\vec{x}^{(0)}\right)\right\}$$

for the current density operator; $\mu_0 = \frac{e\hbar}{2Mc}$ denotes the "Bohr magneton" relative to the mass-value $M$. Hence, the magnetic moment $\vec{M} = \frac{1}{2}\int \vec{x} \wedge \vec{I}\, dv$ takes the form*

$$\vec{M} = \frac{e}{2c}\varrho_3 \vec{x}^{(0)} \wedge \vec{v}^{(0)} + \mu_0 \varrho_3 \vec{\sigma} + \frac{d}{c\,dt}\left(\frac{1}{2}\mu_0 \vec{x}^{(0)} \wedge \varrho_2 \vec{\sigma}\right). \quad (45)$$

The first term represents the magnetic moment arising from the motion of the charge ("orbital" moment), the second is the intrinsic magnetic moment connected with the spin. The third term is in any case negligible for particles in a nucleus since it is of higher order in the velocities.

For "free" vector mesons in an electromagnetic field (the word "free" indicating the absence of nuclear sources), a quite analogous decomposition of the current density and magnetic moment is possible. This has been shown by Proca [21], and somewhat more elaborately by Kemmer [22] from the point of view of the "particle aspect" of meson theory. We repeat Kemmer's derivation by means of the formalism of meson field theory. It will here be convenient to make use of a four-dimensional notation, putting

$$\partial = \frac{\partial}{\partial x^i} - \frac{e}{\hbar c}\boldsymbol{A}_i \wedge \quad (46)$$

and

$$\boldsymbol{G}_{ik} = \partial_i \boldsymbol{U}_k - \partial_k \boldsymbol{U}_i; \quad (47)$$

thus, we have

$$\left.\begin{aligned} (\boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3) &= \vec{\boldsymbol{U}},\ \boldsymbol{U}_4 = -\boldsymbol{U}^4 = -\hat{\boldsymbol{V}}, \\ (\boldsymbol{G}_{23}, \boldsymbol{G}_{31}, \boldsymbol{G}_{12}) &= \hat{\vec{\boldsymbol{G}}},\ (\boldsymbol{G}_{14}, \boldsymbol{G}_{24}, \boldsymbol{G}_{34}) = -(\boldsymbol{G}^{14}, \boldsymbol{G}^{24}, \boldsymbol{G}^{34}) = \vec{\boldsymbol{F}}. \end{aligned}\right\} \quad (48)$$

The part of the current density (6) corresponding to the vector mesons takes the form

$$\hat{I}^{\text{vect}} = -\frac{e}{\hbar c}\left\{\boldsymbol{G}_{ik} \wedge \boldsymbol{U}^k\right\}_3$$

or, according to** (47),

* Use is here made of the formula $\int \vec{x} \wedge \operatorname{rot} \vec{u}\, dv = 2\int \vec{u}\, dv +$ surface integral.

** Use is also made of the relation, easily derived from (46),

$$\partial_i(\boldsymbol{a} \wedge \boldsymbol{b}) = (\partial_i \boldsymbol{a}) \wedge \boldsymbol{b} + \boldsymbol{a} \wedge (\partial_i \boldsymbol{b}).$$

$$\left.\begin{aligned} \hat{I}_i^{\text{vect}} &= -\frac{e}{\hbar c}\left\{\left(\partial_i \boldsymbol{U}_k - \partial_k \boldsymbol{U}_i\right) \wedge \boldsymbol{U}^k\right\}_3 \\ &= \frac{e}{\hbar c}\left\{\boldsymbol{U}^k \wedge \partial_i \boldsymbol{U}_k\right\}_3 + \frac{e}{\hbar c}\left\{\partial_k\left(\boldsymbol{U}_i \wedge \boldsymbol{U}^k\right)\right\}_3 \\ &\quad - \frac{e}{\hbar c}\left\{\boldsymbol{U}_i \wedge \partial_k \boldsymbol{U}^k\right\}_3 . \end{aligned}\right\} \tag{49}$$

From the field equations

$$\kappa^2 \boldsymbol{U}^i = -\partial_k \boldsymbol{G}^{ik} \tag{50}$$

we deduce

$$\kappa^2 \partial_i \boldsymbol{U}^i = -\partial_i \partial_k \boldsymbol{G}^{ik} = -\frac{1}{2}\left(\partial_i \partial_k - \partial_k \partial_i\right) \boldsymbol{G}^{ik};$$

now, by (46),

$$\partial_i \partial_k - \partial_k \partial_i = -\frac{e}{\hbar c} \boldsymbol{F}_{ik} \wedge; \quad \left(\boldsymbol{F}_{ik} = \frac{\partial \boldsymbol{A}_k}{\partial x^i} - \frac{\partial \boldsymbol{A}_i}{\partial x^k}\right) \tag{51}$$

so that

$$\kappa^2 \partial_i \boldsymbol{U}^i = \frac{e}{2\hbar c} \boldsymbol{F}_{ik} \wedge \boldsymbol{G}^{ik} \tag{52}$$

and formula (49) becomes*

$$\left.\begin{aligned} \hat{I}_i^{\text{vect}} &= \frac{e}{\hbar c}\left\{\boldsymbol{U}^k \wedge \partial_i \boldsymbol{U}_k\right\}_3 + \frac{e}{\hbar c}\frac{\partial}{\partial x^k}\left\{\boldsymbol{U}_i \wedge \boldsymbol{U}^k\right\}_3 \\ &\quad - \frac{1}{2}\left(\frac{e}{\kappa \hbar c}\right)^2 \left\{\boldsymbol{U}_i \wedge \left(\boldsymbol{F}_{kl} \wedge \boldsymbol{G}^{kl}\right)\right\}_3 . \end{aligned}\right\} \tag{53}$$

The first term represents the ordinary convection current density, including the dependency on the vector potential; the second term, according to its form, is connected with an intrinsic (magnetic and electric) polarization of the vector mesons, while the third arises from a peculiar polarizability of these mesons in an electromagnetic field. It is noteworthy that the intrinsic current density by itself has a vanishing four-divergence. The current density of the pseudoscalar mesons contains, of course, only a convection term which may be written**

* We notice that $\quad \left\{\partial_i \boldsymbol{K}\right\}_3 = \dfrac{\partial}{\partial x^i} K_3 .$

** We have, according to formula (27) of Part I,

$$a^\dagger a \equiv |a|^2 = \frac{1}{2}\left(a_1^2 + a_2^2\right).$$

$$\vec{\hat{I}}^{\text{ps-scal}} = \{\boldsymbol{\Psi} \wedge \operatorname{grad} \boldsymbol{\Psi}\}_3 - 2\left(\frac{e}{\hbar c}\right)^2 \vec{A} \, |\,\psi\,|^2. \tag{54}$$

According to (48), we may write more explicitly for the total current density* (53) and (54)

$$\left.\begin{aligned} \vec{\hat{I}} = {} & \frac{e}{\hbar c}\left\{\sum_{i=1}^{3} \boldsymbol{U}^i \wedge \operatorname{grad} \boldsymbol{U}^i - \hat{\boldsymbol{V}} \wedge \operatorname{grad} \hat{\boldsymbol{V}} + \boldsymbol{\Psi} \wedge \operatorname{grad} \boldsymbol{\Psi}\right\}_3 \\ & - 2\left(\frac{e}{\hbar c}\right)^2 \vec{A}\left[\,|\,\vec{U}\,|^2 - |\,\hat{V}\,|^2 + |\,\psi\,|^2\right] \\ & + \frac{e}{2\hbar c}\operatorname{rot}\left\{\vec{\boldsymbol{U}} \overset{\wedge}{\wedge} \vec{\boldsymbol{U}}\right\}_3 + \frac{e}{\hbar c}\frac{\partial}{c\,\partial}\left\{\vec{\boldsymbol{U}} \wedge \hat{\boldsymbol{V}}\right\}_3 \\ & - \frac{1}{2}\left(\frac{e}{M_m c^2}\right)^2\left[\vec{U}_1\left(\vec{H}\vec{\hat{G}}_1 - \vec{E}\vec{F}_1\right) + \vec{U}_2\left(\vec{H}\vec{\hat{G}}_2 - \vec{E}\vec{F}_2\right)\right]. \end{aligned}\right\} \tag{55}$$

The magnetic moment is, therefore, (with $\vec{\mathfrak{M}} \equiv \vec{x} \wedge \operatorname{grad}$)

$$\left.\begin{aligned} \vec{M}_{\text{free mes}} = {} & \frac{e}{2\hbar c}\int\left\{\sum_{i=1}^{3} \boldsymbol{U}^i \wedge \vec{\mathfrak{M}}\,\boldsymbol{U}^i - \hat{\boldsymbol{V}} \wedge \vec{\mathfrak{M}}\,\hat{\boldsymbol{V}} + \boldsymbol{\Psi} \wedge \vec{\mathfrak{M}}\,\boldsymbol{\Psi}\right\}_3 dv \\ & - 2\left(\frac{e}{\hbar c}\right)^2 \int \left(\vec{x} \wedge \vec{A}\right)\left[\,|\,\vec{U}\,|^2 - |\,\hat{V}\,|^2 + |\,\psi\,|^2\right] dv \\ & + \frac{e}{2\hbar c}\int\left\{\vec{\boldsymbol{U}} \overset{\wedge}{\wedge} \vec{\boldsymbol{U}}\right\}_3 dv + \frac{e}{2\hbar c}\frac{d}{c\,dt}\int \vec{x} \wedge \left\{\vec{\boldsymbol{U}} \wedge \hat{\boldsymbol{V}}\right\}_3 dv \\ & - \frac{1}{4}\left(\frac{e}{M_m c^2}\right)^2 \int \left[\left(\vec{x} \wedge \vec{U}_1\right)\left(\vec{H}\vec{\hat{G}}_1 - \vec{E}\vec{F}_1\right)\right. \\ & \left. + \left(\vec{x} \wedge \vec{U}_2\right)\left(\vec{H}\vec{\hat{G}}_2 - \vec{E}\vec{F}_2\right)\right] dv. \end{aligned}\right\} \tag{56}$$

The two first lines constitute the "orbital" moment, the third line contains the "intrinsic" moment

$$\vec{M}_{\text{intr}} = \frac{e}{2\hbar c}\int\left\{\vec{\boldsymbol{U}} \overset{\wedge}{\wedge} \vec{\boldsymbol{U}}\right\}_3 dv \tag{57}$$

and a term involving the commutator of a certain expression with the energy of the free meson field; the last term corresponds

* A still more compendious derivation of this expression in a form similar to (49), but directly embodying also the contribution from the pseudoscalar field, is of course possible with the help of the five-dimensional projective formalism of meson theory.

to an "induced" polarization which we henceforth shall neglect. Comparing the expression (57) of the intrinsic magnetic moment with that of the spin angular momentum (cf. NF (60))

$$-\frac{1}{c}\int \vec{\boldsymbol{U}} \wedge \vec{\boldsymbol{F}}\, dv,$$

a glance at the formula (A 4) of the Appendix will show that, for a transverse* vector meson of given wave-number $\vec{k}$ and given electric charge, the expectation value of the magnetic moment is

$$\pm\frac{e}{2\,\hbar}\cdot\frac{1}{\sqrt{k^2+\kappa^2}}$$

times that of the spin; for slow mesons, this becomes $\pm\frac{e}{2\,\hbar\kappa}$ $=\pm\frac{e}{2\,M_m c}$, *i. e.* the normal value with a "Landé factor" unity [23].

Turning now to the expression (43) of the current density of our system of nucleons and mesons, we shall — leaving aside the contributions from the field-dependent parts $\vec{I}_{(1)}+\vec{I}_{(2)}$ — set up the expression of the magnetic dipole moment of the system. In the first place, we see that the term $\vec{I}_{\text{nucl}}$ contributes to this moment a sum of terms of the form (45) $\left(\text{each multiplied by the corresponding factor } \frac{1-\tau_3}{2}\right)$, while $\vec{I}_{\text{free mes}}$ gives rise to the contribution (56). Here, the symbol $\frac{d}{c\,dt}$ should, strictly speaking, be understood as indicating $(i\hbar)^{-1}$ times the commutator with the total energy of free nucleons and mesons; but we may with a negligible error replace this by the total energy of the system, including the couplings (of which the most important is the static interaction between the nucleons), since the modifications so introduced involve both the source constants and the nuclear velocities. The terms of the magnetic moment involving $\frac{d}{c\,dt}$ will then not give any contribution either to the expectation value of the moment in a stationary state of the system or to the

* For a longitudinal meson, the expectation value of spin and magnetic moment is zero. This is also the case for a linearly polarized transverse meson. A well-defined value of the spin ($\pm\hbar$) is obtained for circularly polarized mesons.

matrix elements pertaining to processes in which the total energy is conserved.

From the last term $\vec{I}_{(0)} = \vec{I}_{\text{exch}} + \vec{I}_{\text{prop}}$ of (43) we derive an "exchange magnetic moment" $\vec{M}_{\text{exch}}$ and a further contribution $\vec{M}_{\text{prop}}$ to the magnetic moments of the single nucleons, which we shall call "proper magnetic moment" [24]. The evaluation of

$$\vec{M}_{(0)} = \frac{1}{2} \int \vec{x} \wedge \vec{I}_{(0)} \, dv \tag{58}$$

is carried out in the Appendix, the contributions from the vector and from the pseudoscalar meson fields being calculated separately; it again appears that the resulting expression of $\vec{M}_{\text{exch}}$ is considerably simpler in the mixed theory than in any other*. Using the notation

$$r^{(ik)} = \left| \vec{x}^{(i)} - \vec{x}^{(k)} \right|, \quad \vec{x}_0^{(ik)} = \frac{\vec{x}^{(i)} - \vec{x}^{(k)}}{r^{(ik)}}, \tag{59}$$

it may be written

$$\left.\begin{aligned} \vec{M}_{\text{exch}} = \frac{e}{4\hbar c} \sum_{i \neq k} \left\{ \boldsymbol{\tau}^{(i)} \wedge \boldsymbol{\tau}^{(k)} \right\}_3 & \left\{ \left( \frac{g_2}{\kappa} \right)^2 \left( \vec{\sigma}^{(i)} \wedge \vec{\sigma}^{(k)} \right) \left( 1 - 2\kappa r^{(ik)} \right) \right. \\ & + \left( \frac{g_2}{\kappa} \right)^2 \left[ \left( \vec{\sigma}^{(i)} \wedge \vec{\sigma}^{(k)} \right) \cdot \vec{x}_0^{(ik)} \right] \vec{x}_0^{(ik)} \left( 1 + \kappa r^{(ik)} \right) \\ & \left. - \left( g_1^2 + g_2^2 \vec{\sigma}^{(i)} \vec{\sigma}^{(k)} \right) \left( \vec{x}^{(i)} \wedge \vec{x}^{(k)} \right) \right\} \varphi \left( r^{(ik)} \right). \end{aligned}\right\} \tag{60}$$

The proper magnetic moment is simply the sum of the contributions from the vector and the pseudoscalar meson fields:

$$\vec{M}_{\text{prop}} = \vec{M}_{\text{prop}}^{\text{vect}} + \vec{M}_{\text{prop}}^{\text{ps-scal}}; \tag{61}$$

the latter are

$$\left.\begin{aligned} \vec{M}_{\text{prop}}^{\text{vect}} &= \frac{e}{4\hbar c} \sum_i \int \left\{ \vec{\boldsymbol{S}}^{(i)} \left( \vec{x}' \right) \wedge \vec{\boldsymbol{S}}^{(i)} \left( \vec{x}'' \right) \right\}_3 \left( 1 - \kappa \left| \vec{x}' - \vec{x}'' \right| \right) \varphi \, dv' \, dv'' \\ \vec{M}_{\text{prop}}^{\text{ps-scal}} &= -\frac{e}{4\hbar c} \sum_i \int \left\{ \left\{ \vec{\boldsymbol{P}}^{(i)} \left( \vec{x}' \right) \wedge \vec{\boldsymbol{P}}^{(i)} \left( \vec{x}'' \right) \right\}_3 \kappa \left| \vec{x}' - \vec{x}'' \right| \right. \\ & \quad \left. - \left[ \left\{ \vec{\boldsymbol{P}}^{(i)} \left( \vec{x}' \right) \wedge \vec{\boldsymbol{P}}^{(i)} \left( \vec{x}'' \right) \right\}_3 \cdot \vec{x}_0 \right] \vec{x}_0 \left( 1 + \kappa \left| \vec{x}' - \vec{x}'' \right| \right) \right\} \varphi \, dv' \, dv'', \end{aligned}\right\} \tag{62}$$

* It should be noted that the simplification occurs in the part of $\vec{M}_{\text{exch}}$ which is not translation invariant (cf. the Appendix), *i. e.* directly results from the simpler form of $\vec{J}$ on mixed theory.

with $\vec{x}_0 \equiv \dfrac{\vec{x}' - \vec{x}''}{\left|\vec{x}' - \vec{x}''\right|}$. In these formulae, $\vec{\boldsymbol{S}}^{(i)}$ and $\vec{\boldsymbol{P}}^{(i)}$ are the contributions of the $i$-th nucleon to the source densities $\vec{\boldsymbol{S}}$ and $\vec{\boldsymbol{P}}$ (cf. NF, p. 10).

According to NF (4, 24) we may write

$$\left.\begin{aligned} \vec{\boldsymbol{S}}^{(i)}\left(\vec{x}\right) &= \frac{g_2}{\kappa}\,\boldsymbol{\tau}^{(i)}\,\varrho_3^{(i)}\,\vec{\sigma}^{(i)}\,D\left(\vec{x}-\vec{x}^{(i)}\right) \\ \vec{\boldsymbol{P}}^{(i)}\left(\vec{x}\right) &= \frac{f_2}{\kappa}\,\boldsymbol{\tau}^{(i)}\,\vec{\sigma}^{(i)}\,D\left(\vec{x}-\vec{x}^{(i)}\right), \end{aligned}\right\} \tag{64}$$

the delta-function being replaced by a continuous distribution function $D\left(\vec{x}-\vec{x}^{(i)}\right)$ $\left(\text{with } \int D\left(\vec{x}\right) dv = 1\right)$. The formulae (62) and (63) then take the simpler form

$$\vec{M}_{\text{prop}} = -\mu_0\,\mu \sum_i \tau_3^{(i)}\,\vec{\sigma}^{(i)}, \tag{65}$$

where $\mu_0$ denotes (as above) the nuclear magneton, while

$$\mu^{\text{vect}} = \frac{g_2^2}{\hbar c}\cdot\frac{M}{M_m}\cdot\frac{2}{\kappa}\int D\left(\vec{x}'\right) D\left(\vec{x}''\right)(1-\kappa\varrho)\,\varphi(\varrho)\,dv'\,dv'' \tag{66}$$

$$\mu^{\text{ps-scal}} = \frac{f_2^2}{\hbar c}\cdot\frac{M}{M_m}\cdot\frac{2}{3\kappa}\int D\left(\vec{x}'\right) D\left(\vec{x}''\right)(1-2\kappa\varrho)\,\varphi(\varrho)\,dv'\,dv'', \tag{67}$$

with $\varrho \equiv \left|\vec{x}' - \vec{x}''\right|$; the last formula is valid on the assumption that the distribution function $D\left(\vec{x}\right)$ is spherically symmetrical. The value of $\mu$ on the mixed theory is therefore (putting $f_2^2 = g_2^2$)

$$\left.\begin{aligned} \mu &= \mu^{\text{vect}} + \mu^{\text{ps-scal}} \\ &= \frac{g_2^2}{\hbar c}\cdot\frac{M}{M_m}\cdot\frac{2}{3\kappa}\int D\left(\vec{x}'\right) D\left(\vec{x}''\right)(4-5\kappa\varrho)\,\varphi(\varrho)\,dv'\,dv''. \end{aligned}\right\} \tag{68}$$

According to (65), we should expect the total magnetic moments of proton (at rest) and neutron to be $\vec{\sigma}\,\mu_0\,\mu_P$ and $\vec{\sigma}\,\mu_0\,\mu_N$, respectively, with

$$\mu_P = 1 + \mu, \qquad \mu_N = -\mu.$$

For a closer discussion of the expressions (66), (67), and (68) for $\mu$, the reader is referred to the forthcoming paper by SERPE: it appears that the mixed theory will under reasonable assumptions for the distribution function $D$ yield a value of $\mu$ of the right sign and the right order of magnitude. From the latest experimental determinations of the magnetic moments of proton [25] and neutron [26],

$$\left.\begin{aligned} \mu_P &= 2{,}7896 \pm 0{,}0008 \\ \mu_N &= -1{,}935 \pm 0{,}02, \end{aligned}\right\} \qquad (70)$$

it would seem, however, that there is a slight difference between the "proper" magnetic moments of the two particles. Such a dissymmetry cannot be accounted for by the present theory, since it only arises in a higher approximation which should be discarded according to our general prescription. The manner in which the dissymmetry appears has been explained by FRÖHLICH, HEITLER and KEMMER [24]: in the expression of the self-energy of a nucleon in an external magnetic field, the second approximation terms consist of quotients of matrix elements independent of the magnetic field by differences of energy values for the initial and the intermediate states considered; now, such differences will contain a term proportional to the magnetic field which, when due account is taken of the magnetic energy $\pm \mu_0 H$ of the free proton states, turns out to be different for the alternative cases of a proton or a neutron. This effect clearly falls outside the scope of our theory* and can only be expected to present itself in a theory yielding a correct treatment of self-energy problems; we may provisorily allow for it by replacing the above expression (65) of $\vec{M}_{\text{prop}}$ by

* It may be pointed out that an asymmetry of the kind here discussed could not be derived from the assumption of a small deviation from the symmetrical KEMMER combination of charged and neutral meson fields. In fact, the most general combination of such fields would be obtained by replacing in the expressions of the source densities each $\tau_i$ by a linear combination of the form $\sum_k a_{ik}\tau_k + b_i$ with c-number coefficients $a_{ik}$, $b_i$. Then, the same combination would also appear in the expressions of the static potentials and fields, and it is immediately apparent from formula (42) that this would not give rise to any term independent of $\tau_3$ in the expression of the proper moment, such as would be required for the asymmetry effect in question.

$$\vec{M}^{\text{exp}}_{\text{prop}} = \mu_0 \sum_i \left\{ \frac{1-\tau_3^{(i)}}{2}(\mu_P - 1) + \frac{1+\tau_3^{(i)}}{2}\mu_N \right\} \vec{\sigma}^{(i)}, \tag{71}$$

where $\mu_P$ and $\mu_N$ have their empirical values (70).

## § 6. Electromagnetic properties of the deuteron.

A few remarks may still be added concerning the application of the preceding formulae to the simplest nuclear system, the deuteron. Comparing the treatment in meson theory with that in the previous theories, in which the proton was treated as a point charge in some arbitrarily assumed short range nuclear field, we see that, as regards the electric properties, there is no other difference than that arising from the influence of the latter field on the form of the wave functions. In fact, as stated in section 3 above, the exchange dipole moment gives only a negligible contribution, while the exchange quadrupole moment with respect to the centre of gravity of the deuteron vanishes according to formula (28). The exchange magnetic moment, on the other hand, does play an appreciable part which, in some cases, may even be quite considerable; examples of this are provided by PAIS' investigation of the photomagnetic effect of the deuteron and the inverse process of neutron capture by protons [7].

Since the exchange terms do not contribute anything to the expectation values of the corresponding quantities in stationary states of the system, no new element is introduced by our theory into the situation with regard to the magnetic moment of the deuteron in relation to its quadrupole moment. The existence of the latter, with a value estimated [27] to be

$$Q = 2{,}73 \cdot 10^{-27}\ \text{cm}^2 \tag{72}$$

with an uncertainty of about 2 %, implies that the ground state of the deuteron must be a mixture of a $^3S$ state with a small amount $\delta$ of $^3D_1$ state. The magnetic moment of the ground state can then easily be calculated in terms of $\delta$. The exact expression of the $z$-component of the magnetic moment is

$$\left.\begin{aligned} M_z = \mu_0 \sum_{i=1,2} \Bigg\{ \left( \frac{1-\tau_3^{(i)}}{2}\mu_P + \frac{1-\tau_3^{(i)}}{2}\mu_N \right) \sigma_z^{(i)} \\ + \frac{1-\tau_3^{(i)}}{2} l_z^{(i)} - \frac{1-\tau_3^{(i)}}{2} \cdot \left(1-\varrho_3^{(i)}\right)\left(l_z^{(i)} + \sigma_z^{(i)}\right) \Bigg\}, \end{aligned}\right\} \tag{73}$$

$l_z^{(i)}$ denoting the $z$-component of the orbital angular momentum of the *i-th* nucleon. For its expectation value* in a stationary state of the nucleus, we get, since the mean value of $\tau_3^{(i)}$ is zero and that of $\varrho_3^{(i)}$ is approximately unity**,

$$\mathfrak{M}\left\{M_z\right\} \approx \mu_0 \left[\left(\mu_P + \mu_N\right) \mathfrak{M}\left\{\frac{1}{2} S_z\right\} + \frac{1}{2} \mathfrak{M}\left\{L_z\right\}\right],$$

where $S_z$, $L_z$ represent the $z$-components of the total spin and total orbital momentum, respectively. Characterizing the stationary state in the usual way by the quantum numbers $L$, $S$, $J$, $m$, this may be written

$$\mathfrak{M}\left\{M_z\right\} \approx \mu_0 \, m \, \mu_D^{(LSJ)},$$

with the Landé factor

$$\begin{aligned}\mu_D^{(LSJ)} = & \left(\mu_P + \mu_N\right) \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} \\ & + \frac{1}{2} \frac{J(J+1) + L(L+1) - S(S+1)}{2J(J+1)}.\end{aligned}$$

For the ground state we thus get

$$\left.\begin{aligned}\mu_D &= (1-\delta)\left(\mu_P + \mu_N\right) + \delta\left[-\frac{1}{2}\left(\mu_P + \mu_N\right) + \frac{3}{4}\right] \\ &= \left(1 - \frac{3}{2}\delta\right)\left(\mu_P + \mu_N\right) + \frac{3}{4}\delta.\end{aligned}\right\} \quad (74)$$

The direct determination of $\mu_D$ [25],

$$\mu_D = 0{,}8565 \pm 0{,}0004, \quad (75)$$

lies very close to $\mu_N + \mu_P$, which, according to (70), is $0{,}85 \pm 0{,}02$; on the other hand, the amount $\delta$ of $D$-state necessary to account for the quadrupole moment (72), while depending on the form of meson theory adopted, is of the order of magnitude of a few percent***. So far, there is thus no certain discrepancy between

* The expectation value of $A$ will be denoted by $\mathfrak{M}\left\{A\right\}$.

** The relativistic correction arising from the factors $\left(1 - \varrho_3^{(i)}\right)$ was discussed by H. Margenau [28].

*** This is also the case, as a rough estimate shows, with the derivation of the quadrupole moment based on the non-static directional coupling of the formula (74) and the empirical results (70), (75) and (72), though the margin of error appears rather strained. A decision on this point can only be arrived at by more precise measurements, especially of the neutron moment. It should also be remembered that a future theory, taking a correct account of the universal limiting length, might, analogously to the dissymmetry effect discussed in section 5, yield a small correction to the purely additive expression (73) of $M_z$ arising from the nuclear field interaction between the neutron and the proton.

mixed theory, as developed in NF, Part III, if the nuclear source constants are chosen so as to give the right order of magnitude for the quadrupole moment. (If, however, the relations between these constants suggested by the five-dimensional formalism [4] are adopted, the quadrupole moment vanishes in that approximation).

# Appendix.

## Mean values for the zero-point meson field.

Let us take as usual progressive plane waves

$$u\left(\vec{k};\vec{x}\right) = L^{-3/2}\, e^{i\vec{k}\vec{x}}, \tag{A 1}$$

satisfying a cyclic condition within a cube of side $L$:

$$\vec{k} = \frac{2\pi}{L}(n_1, n_2, n_3); \quad (n\text{'s integers}) \tag{A 2}$$

the wave-vector $\vec{k}$ is connected with the momentum $\vec{p}$ and energy $E$ of a meson by

$$\left.\begin{aligned} \vec{p} &= \hbar\,\vec{k} \\ E &= \hbar\, c\, \varepsilon_k \\ \text{with } \varepsilon_k &= \sqrt{k^2 + \kappa^2}. \end{aligned}\right\} \tag{A 3}$$

We introduce quantized amplitudes $\boldsymbol{a}\left(\vec{k};j\right)$ referring to mesons of a given wave-vector (momentum and energy); the index $j$ will be given the value 0 for pseudoscalar mesons and the values 1, 2, 3 for vector mesons, 1 and 2 referring to two independent kinds of transversal (linear) polarization of the vector meson field, 3 to the longitudinal polarization. The directions of polarization will be characterized by three mutually orthogonal unit vectors $\vec{e}_j\left(\vec{k}\right)$.

The meson fields may then be expressed as a superposition of such plane waves in the following way:

$$\left.\begin{aligned}
\vec{\boldsymbol{U}} &= \sqrt{\frac{\hbar c}{2}} \sum_{\vec{k}} \frac{1}{\sqrt{\varepsilon_k}} \sum_{j=1}^{3} \vec{e}_j\left(\vec{k}\right) \cdot \left(\frac{\varepsilon_k}{\varkappa}\right)^{\delta_{j3}} \left[\boldsymbol{a}\left(\vec{k}; j\right) u\left(\vec{k}\right) + \boldsymbol{a}^{\dagger}\left(\vec{k}; j\right) u^*\left(\vec{k}\right)\right] \\
\vec{\boldsymbol{F}} &= i\sqrt{\frac{\hbar c}{2}} \sum_{\vec{k}} \sqrt{\varepsilon_k} \sum_{j=1}^{3} \vec{e}_j\left(\vec{k}\right) \cdot \left(\frac{\varkappa}{\varepsilon_k}\right)^{\delta_{j3}} \left[\boldsymbol{a}\left(\vec{k}; j\right) u\left(\vec{k}\right) - \boldsymbol{a}^{\dagger}\left(\vec{k}; j\right) u^*\left(\vec{k}\right)\right], \\
\boldsymbol{\Psi} &= \sqrt{\frac{\hbar c}{2}} \sum_{\vec{k}} \frac{1}{\sqrt{\varepsilon_k}} \left[\boldsymbol{a}\left(\vec{k}; 0\right) u\left(\vec{k}\right) + \boldsymbol{a}^{\dagger}\left(\vec{k}; 0\right) u^*\left(\vec{k}\right)\right] \\
\boldsymbol{\Phi} &= -i\sqrt{\frac{\hbar c}{2}} \sum_{\vec{k}} \sqrt{\varepsilon_k} \left[\boldsymbol{a}\left(\vec{k}; 0\right) u\left(\vec{k}\right) - \boldsymbol{a}^{\dagger}\left(\vec{k}; 0\right) u^*\left(\vec{k}\right)\right].
\end{aligned}\right\} \quad \text{(A 4)}$$

The commutation rules for the amplitudes are

$$\left.\begin{aligned}
&\left[a_{\boldsymbol{n}}\left(\vec{k}; j\right),\ a^{\dagger}_{\boldsymbol{n}'}\left(\vec{k}'; j'\right)\right]_{-} = \delta_{\boldsymbol{nm}}\, \delta\left(\vec{k}, \vec{k}'\right) \delta_{jj'}, \\
&\text{all other pairs commuting;}
\end{aligned}\right\} \quad \text{(A 5)}$$

the operators

$$N_{\boldsymbol{n}}\left(\vec{k}; j\right) = a^{\dagger}_{\boldsymbol{n}}\left(\vec{k}; j\right) a_{\boldsymbol{n}}\left(\vec{k}; j\right) \quad \text{(A 6)}$$

represent the numbers of mesons of kind $j$, wave-vector $\vec{k}$ and character $\boldsymbol{n}$ with respect to the electric charge: $N_3$ is the number of neutral mesons, $N_1 + N_2$ the total number of positively and negatively charged mesons. We get the numbers of positive or negative mesons, separately, when replacing the amplitudes $a_1$, $a_2$ by

$$b_1 = \frac{1}{\sqrt{2}}\left(a_1 + i a_2\right), \quad b_2 = \frac{1}{\sqrt{2}}\left(a_1 - i a_2\right),$$

which obey similar commutation rules; then

$$N'_1 = b^{\dagger}_1 b_1, \quad N'_2 = b^{\dagger}_2 b_2$$

are the numbers of positive and negative mesons, respectively, the total charge of the meson field taking the form

$$\int \varrho_{\text{mes}}\, dv = e \sum_{\vec{k}} \sum_{j=0}^{3} \left[N'_1\left(\vec{k}; j\right) - N'_2\left(\vec{k}; j\right)\right].$$

For the state corresponding to no meson, the above formulae yield, with

$$v\left(\vec{k};\, \vec{x}-\vec{x}'\right) = u\left(\vec{k};\, \vec{x}\right) u^{*}\left(\vec{k};\, \vec{x}'\right) = L^{-3} e^{i\vec{k}(\vec{x}-\vec{x}')},$$

the following mean values:

$$\left.\begin{aligned}
\mathfrak{M}_0\left\{F^{\mu}_{\boldsymbol{n}}\left(\vec{x}\right) F^{\nu}_{\boldsymbol{n}}\left(\vec{x}'\right)\right\} &= \frac{\hbar c}{2}\sum_{\vec{k}} \varepsilon_k \sum_j e^{\mu}_j e^{\nu}_j \left(\frac{\varkappa}{\varepsilon_k}\right)^{2\delta_{j3}} v\left(\vec{k};\, \vec{x}-\vec{x}'\right)\\
&= \frac{\hbar c}{2}\sum_{\vec{k}} \varepsilon_k \left[\delta^{\mu\nu} + \frac{k^{\mu} k^{\nu}}{k^2}\left(\frac{\varkappa^2}{\varepsilon_k^2}-1\right)\right] v\left(\vec{k};\, \vec{x}-\vec{x}'\right)\\
&= \frac{\hbar c}{2}\sum_{\vec{k}} \varepsilon_k \left(\delta^{\mu\nu} + \frac{k^{\mu} k^{\nu}}{\varepsilon_k^2}\right) v\left(\vec{k};\, \vec{x}-\vec{x}'\right),\\
\mathfrak{M}_0\left\{U^{\mu}_{\boldsymbol{n}}\left(\vec{x}\right) U^{\nu}_{\boldsymbol{n}}\left(\vec{x}'\right)\right\} &= \frac{\hbar c}{2}\sum_{\vec{k}} \frac{1}{\varepsilon_k} \left[\delta^{\mu\nu} + \frac{k^{\mu} k^{\nu}}{k^2}\left(\frac{\varepsilon_k^2}{\varkappa^2}-1\right)\right] v\left(\vec{k};\, \vec{x}-\vec{x}'\right)\\
&= \frac{\hbar c}{2}\sum_{\vec{k}} \frac{1}{\varepsilon_k} \left(\delta^{\mu\nu} - \frac{k^{\mu} k^{\nu}}{\varkappa^2}\right) v\left(\vec{k};\, \vec{x}-\vec{x}'\right),\\
\mathfrak{M}_0\left\{\Phi_{\boldsymbol{n}}\left(\vec{x}\right) \Phi_{\boldsymbol{n}}\left(\vec{x}'\right)\right\} &= \frac{\hbar c}{2}\sum_{\vec{k}} \varepsilon_k\, v\left(\vec{k};\, \vec{x}-\vec{x}'\right);
\end{aligned}\right\} \tag{A 7}$$

from the canonical commutation rules for $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{F}}$, it may further be deduced without calculation:

$$\mathfrak{M}_0\left\{U^{\mu}_{\boldsymbol{n}}\left(\vec{x}\right) F^{\nu}_{\boldsymbol{n}}\left(\vec{x}'\right)\right\} = -\mathfrak{M}_0\left\{F^{\mu}_{\boldsymbol{n}}\left(\vec{x}\right) U^{\nu}_{\boldsymbol{n}}\left(\vec{x}'\right)\right\} = \frac{\hbar c}{2i}\,\delta^{\mu\nu}\,\delta\left(\vec{x}-\vec{x}'\right); \tag{A 8}$$

all other mean values of products of two field components vanish. The summations (or, in the limit $L \to \infty$, integrations) over the wave-vector components occurring in the preceding formulae may all be derived from the relation

$$K_0(\varkappa r) = 2\pi^2 \lim_{L\to\infty} \sum_{\vec{k}} \frac{1}{\varepsilon_k^3}\, v\left(\vec{k};\, \vec{x}+\vec{x}'\right), \quad \left(r \equiv \left|\vec{x}-\vec{x}'\right|\right) \tag{A 9}$$

where $K_0(z)$ is a Bessel function of the second kind with imaginary argument, connected with Hankel's functions by the general formula*

* For the definition and properties of $K_n(z)$, see G. WATSON, A treatise on the theory of Bessel functions (1922), *3·7* (p. 77), *3·71* (p. 79), *6·16* (p. 172). In E. WHITTAKER and G. WATSON, A course of modern analysis, *17·71* (4th Ed., p. 373), a function called $K_n(z)$, but differing from the one here used by a factor $(-1)^n$, is considered.

$$K_n(z) = \frac{\pi i}{2} e^{n\frac{\pi i}{2}} H_n^{(1)}(iz).$$

In fact, since

$$(\kappa^2 - \Delta)\, v = \varepsilon_k^2\, v,$$

we get

$$\left.\begin{aligned} \frac{\hbar c}{2} \sum_{\vec{k}} \frac{1}{\varepsilon_k} v\left(\vec{k};\, \vec{x} - \vec{x}'\right) &= \frac{\hbar c}{4\pi^2} (\kappa^2 - \Delta)\, K_0(\kappa r) \\ &= \frac{\hbar c}{4\pi^2} \cdot \frac{\kappa}{r} K_1(\kappa r) \equiv \mathcal{E}(r), \\ \frac{\hbar c}{2} \sum_{\vec{k}} \varepsilon_k\, v\left(\vec{k};\, \vec{x} - \vec{x}'\right) &= (\kappa^2 - \Delta)\, \mathcal{E}(r) \equiv \mathcal{D}(r); \end{aligned}\right\} \quad (\text{A } 10)$$

the formulae (A 7) may then be written as follows:

$$\left.\begin{aligned} \mathfrak{M}_0\left\{F_{\boldsymbol{n}}^{\mu}\left(\vec{x}\right) F_{\boldsymbol{n}}^{\nu}\left(\vec{x}'\right)\right\} &= \delta^{\mu\nu}\, \mathcal{D}(r) + \frac{\partial^2}{\partial x^{\mu}\, \partial x^{\nu}}\, \mathcal{E}(r) \\ \mathfrak{M}_0\left\{U_{\boldsymbol{n}}^{\mu}\left(\vec{x}\right) U_{\boldsymbol{n}}^{\nu}\left(\vec{x}'\right)\right\} &= \delta^{\mu\nu}\, \mathcal{E}(r) - \frac{1}{\kappa^2} \frac{\partial^2}{\partial x^{\mu}\, \partial x^{\nu}}\, \mathcal{E}(r) \\ \mathfrak{M}_0\left\{\Phi_{\boldsymbol{n}}\left(\vec{x}\right) \Phi_{\boldsymbol{n}}\left(\vec{x}'\right)\right\} &= \mathcal{D}(r). \end{aligned}\right\} \quad (\text{A } 11)$$

We are interested in

$$\left.\begin{aligned} \mathfrak{M}_1 &= \int dv'\, \mathfrak{M}_0\left\{\left(\vec{f}^{(i)}\, \vec{U}_{\mathbf{1}}\right)_{\vec{x}} \left(\vec{f}^{(i)}\, \vec{U}_{\mathbf{1}}\right)_{\vec{x}'}\right\} \\ \mathfrak{M}_2 &= \int dv'\, \mathfrak{M}_0\left\{\left(\vec{f}^{(i)} \wedge \vec{F}_{\mathbf{1}}\right)_{\vec{x}} \left(\vec{f}^{(i)} \wedge \vec{F}_{\mathbf{1}}\right)_{\vec{x}'}\right\} \\ \mathfrak{M}_3 &= \int dv'\, \vec{f}^{(i)}\, \vec{f}^{(i)\prime}\, \mathfrak{M}_0\left\{\Phi_{\boldsymbol{n}}\, \Phi_{\boldsymbol{n}}'\right\}, \end{aligned}\right\} \quad (\text{A } 12)$$

where

$$\vec{f}^{(i)}\left(\vec{x}\right) = \operatorname{grad}^{(i)} \varphi\left(\left|\vec{x} - \vec{x}^{(i)}\right|\right) = -\operatorname{grad} \varphi.$$

From (A 11) we get

Formula (A 9), after the summation over $\vec{k}$ on the right has been replaced in the limit by $\frac{L^3}{8\pi^3}\int \cdots\, dk_x\, dk_y\, dk_z$, or $\frac{L^3}{8\pi^3}\int \cdots\, k^2\, dk\, d\Omega$, and the angle integration performed, takes the form

$$K_0(\kappa r) = \frac{1}{r}\int_0^{\infty} \frac{\sin kr \cdot k\, dk}{(k^2 + \kappa^2)^{3/2}};$$

this last relation is in its turn readily derived from Basset's formula (see G. Watson, loc. cit. *6·16* (p. 172))

$$K_0(\kappa r) = \int_0^{\infty} \frac{\cos k\, dk}{\sqrt{k^2 + (\kappa r)^2}} = \int_0^{\infty} \frac{\cos kr\, dk}{\sqrt{k^2 + \kappa^2}}$$

by a partial integration.

$$\left\{\begin{aligned}
\mathfrak{M}_1 &= \int dv' \vec{f}^{(i)} \vec{f}^{(i)\prime}\, \mathcal{E} - \frac{1}{\kappa^2} \int dv' \sum_{\mu,\nu} f^{(i)\mu}\left(\vec{x}\right) f^{(i)\nu}\left(\vec{x}'\right) \frac{\partial^2}{\partial x^\mu\, \partial x^\nu}\, \mathcal{E} \\
\mathfrak{M}_2 &= \int dv' \vec{f}^{(i)} \vec{f}^{(i)\prime} (2\, \mathcal{D} + \Delta\, \mathcal{E}) - \int dv' \sum_{\mu,\nu} f^{(i)\mu}\left(\vec{x}\right) f^{(i)\nu}\left(\vec{x}'\right) \frac{\partial^2}{\partial x^\mu\, \partial x^\nu}\, \mathcal{E} \\
\mathfrak{M}_3 &= \int dv' \vec{f}^{(i)} \vec{f}^{(i)\prime}\, \mathcal{D}\, .
\end{aligned}\right.$$

By partial integrations, the last integral in $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is easily transformed into

$$\int dv' \vec{f}^{(i)} \vec{f}^{(i)\prime} \Delta\, \mathcal{E}\, ,$$

so that, taking account of (A 10), we get

$$\mathfrak{M}_1 = \kappa^{-2}\, \mathfrak{M}_3\, , \quad \mathfrak{M}_2 = 2\, \mathfrak{M}_3\, . \tag{A 13}$$

Further, since $\varphi\left(\left|\vec{x} - \vec{x}^{(i)}\right|\right)$ satisfies the equation

$$(\kappa^2 - \Delta)\, \varphi = \delta\left(\vec{x} - \vec{x}^{(i)}\right),$$

we may write

$$\begin{aligned}
\mathfrak{M}_3 &= \vec{f}^{(i)} \cdot \mathrm{grad}^{(i)} \int dv'\, \varphi\left(\left|\vec{x}' - \vec{x}^{(i)}\right|\right) (\kappa^2 - \Delta')\, \mathcal{E} \\
&= \vec{f}^{(i)} \cdot \mathrm{grad}^{(i)}\, \mathcal{E}\left(\left|\vec{x} - \vec{x}^{(i)}\right|\right) \\
&= \frac{d\varphi\left(r^{(i)}\right)}{d r^{(i)}} \cdot \frac{d\, \mathcal{E}\left(r^{(i)}\right)}{d r^{(i)}}
\end{aligned}$$

with $r^{(i)} \equiv \left|\vec{x} - \vec{x}^{(i)}\right|$. From (A 10) and a recurrence formula for the $K_n$, we finally obtain

$$\left.\begin{aligned}
\mathfrak{M}_3 &= -\frac{d\varphi\left(r^{(i)}\right)}{d r^{(i)}} \cdot \frac{\hbar c}{4\pi^2} \cdot \frac{\kappa^2}{r^{(i)}} K_2\left(\kappa r^{(i)}\right) \\
&= \frac{e^{-\kappa r^{(i)}}}{4\pi r^{(i)}} \left(\kappa + \frac{1}{r^{(i)}}\right) \frac{\hbar c}{4\pi^2} \cdot \frac{\kappa^2}{r^{(i)}} K_2\left(\kappa r^{(i)}\right).
\end{aligned}\right\} \tag{A 14}$$

## The exchange and proper current density and related quantities.

We shall here be concerned with the calculation of

$$\int \vec{I}_{(0)}\, dv \quad \text{and} \quad \vec{M}_{(0)} = \frac{1}{2} \int \vec{x} \wedge \vec{I}_{(0)}\, dv\, ,$$

$\vec{I}_{(0)}$ being given by (42). For the evaluation of the first integral, we need formula NF (89), which we re-write in the form

$$\psi\left(\left|\vec{x}-\vec{x}'\right|\right) = \int \varphi\left(\left|\vec{x}-\vec{x}''\right|\right)\varphi\left(\left|\vec{x}'-\vec{x}''\right|\right) dv'' = \frac{e^{-\kappa\left|\vec{x}-\vec{x}'\right|}}{8\pi\kappa}; \quad \text{(A 15)}$$

from this, we deduce

$$\operatorname{grad}\psi\left(\left|\vec{x}-\vec{x}'\right|\right) = -\frac{1}{2}\left(\vec{x}-\vec{x}'\right)\varphi\left(\left|\vec{x}-\vec{x}'\right|\right) \quad \text{(A 16)}$$

and $\left(\text{since } \Delta\varphi - \kappa^2\varphi = -\delta\left(\vec{x}-\vec{x}'\right)\right)$

$$\Delta\psi - \kappa^2\psi = -\varphi\left(\left|\vec{x}-\vec{x}'\right|\right). \quad \text{(A 17)}$$

On account of the symmetrization bars in formula (42), we may in all following calculations freely interchange the order of factors, even though they are not commutable; for simplicity, we shall omit the bars from now on. From NF (9, 14) we get, in the first place,

$$\vec{\boldsymbol{F}}^{\circ} \wedge \boldsymbol{V}^{\circ} = \int \boldsymbol{N}' \wedge \boldsymbol{N}'' \operatorname{grad}' \varphi\left(\left|\vec{x}-\vec{x}'\right|\right)\varphi\left(\left|\vec{x}-\vec{x}''\right|\right) dv'\, dv'', \quad \text{(A 18)}$$

whence, by (A 15),

$$\left.\begin{aligned}\int \vec{\boldsymbol{F}}^{\circ} \wedge \boldsymbol{V}^{\circ} dv &= \int \boldsymbol{N}' \wedge \boldsymbol{N}'' \operatorname{grad}' \psi\left(\left|\vec{x}'-\vec{x}''\right|\right) dv'\, dv'' \\ &= \frac{1}{2}\int \boldsymbol{N}' \wedge \boldsymbol{N}\left(\vec{x}-\vec{x}'\right)\varphi\left(\left|\vec{x}-\vec{x}'\right|\right) dv\, dv',\end{aligned}\right\} \text{(A 19)}$$

by (A 16). The $\mathbf{3}$-component of this quantity, multiplied by $-\frac{e}{\hbar c}$, coincides with the first term of the expression $\vec{J}$ given by (31).

Similarly, we get from NF (14)

$$\begin{aligned}\vec{\boldsymbol{G}}^{\circ} \overset{\wedge}{\wedge} \vec{\boldsymbol{U}}^{\circ} &= -\int \vec{\boldsymbol{G}}^{\circ} \overset{\wedge}{\wedge} \left(\vec{\boldsymbol{S}}' \wedge \operatorname{grad}' \varphi\right) dv' \\ &= -\int \left(\vec{\boldsymbol{S}}' \wedge \vec{\boldsymbol{G}}^{\circ}\right) \operatorname{grad}' \varphi\, dv' + \int \vec{\boldsymbol{S}}' \wedge \left(\vec{\boldsymbol{G}}^{\circ} \operatorname{grad}' \varphi\right) dv';\end{aligned}$$

inserting in the first term the value NF (37) of $\vec{\boldsymbol{G}}^{\circ}$ and in the second its value NF (10), we have

$$\left.\begin{aligned}\vec{G}^{\circ}\underset{\wedge}{\wedge}\vec{U}^{\circ} = &-\kappa^2\int\left(\vec{S}'\wedge\vec{S}''\right)\operatorname{grad}'\varphi\left(\left|\vec{x}-\vec{x}'\right|\right)\varphi\left(\left|\vec{x}-\vec{x}''\right|\right)dv'\,dv''\\ &-\int\left[\left(\vec{S}''\operatorname{grad}''\right)\wedge\left(\vec{S}'\operatorname{grad}''\right)\right]\\ &\qquad\operatorname{grad}'\varphi\left(\left|\vec{x}-\vec{x}'\right|\right)\varphi\left(\left|\vec{x}-\vec{x}''\right|\right)dv'\,dv''\\ &+\int\vec{S}'\wedge\left(\vec{S}\operatorname{grad}'\varphi\right)dv'-\int\vec{S}'\wedge\operatorname{div}\left(\operatorname{grad}'\varphi\wedge\vec{U}^{\circ}\right)dv',\end{aligned}\right\}\text{(A 20)}$$

since $\operatorname{rot}\vec{U}^{\circ}\cdot\operatorname{grad}'\varphi = -\operatorname{div}\left(\operatorname{grad}'\varphi\wedge\vec{U}^{\circ}\right)$. Again using A (15) and A (16), we obtain immediately

$$\left.\begin{aligned}\int\vec{G}^{\circ}\underset{\wedge}{\wedge}\vec{U}^{\circ}\,dv = &-\frac{1}{2}\int\kappa^2\left(\vec{S}'\wedge\vec{S}\right)\left(\vec{x}-\vec{x}'\right)\varphi\left(\left|\vec{x}-\vec{x}'\right|\right)dv\,dv'\\ &+\int\Big\{\left[\left(\vec{S}'\operatorname{grad}'\right)\wedge\left(\vec{S}\operatorname{grad}\right)\right]\operatorname{grad}\psi\\ &-\vec{S}'\wedge\left(\vec{S}\operatorname{grad}\varphi\right)\Big\}dv\,dv',\end{aligned}\right\}\text{(A 21)}$$

the integral over the last term of (A 20) reducing to a vanishing surface integral. The first term of (A 21), by taking the **3**-component and multiplying by $\frac{e}{\hbar c}$, gives just the remaining term of the expression (31) of $\vec{J}$. The last term of (A 21) would subsist in the integral exchange current on a pure vector meson theory; in the mixed theory, however, it is compensated by the contribution from the pseudoscalar field.

In fact, from NF (30, 40) we have

$$\left.\begin{aligned}\vec{\Gamma}^{\circ}\wedge\Psi^{\circ} = &\int\vec{P}\wedge\left(\vec{P}'\operatorname{grad}'\varphi\right)dv'\\ &+\int\left[\left(\vec{P}''\operatorname{grad}''\right)\wedge\left(\vec{P}'\operatorname{grad}'\right)\right]\operatorname{grad}''\varphi\left(\left|\vec{x}-\vec{x}'\right|\right)\varphi\left(\left|\vec{x}-\vec{x}''\right|\right)dv'\,dv'',\end{aligned}\right\}\text{(A 22)}$$

whence, by (A 15),

$$\left.\begin{aligned}\int\vec{\Gamma}^{\circ}\wedge\Psi^{\circ}\,dv = &\int\Big\{\vec{P}'\wedge\left(\vec{P}\operatorname{grad}\varphi\right)\\ &-\left[\left(\vec{P}'\operatorname{grad}'\right)\wedge\left(\vec{P}\operatorname{grad}\right)\right]\operatorname{grad}\psi\Big\}dv\,dv'.\end{aligned}\right\}\text{(A 23)}$$

In such an expression we may indeed, on the mixed theory, substitute $\vec{S}$ for $\vec{P}$, apart from terms of higher order in the nuclear velocities. Summing up, we have thus established formula (44).

We now turn to the calculation of $\vec{M}_{(0)}$, which runs entirely parallel to the preceding derivation of (44). Only, the role of the auxiliary relations (A 15), (A 16) is in this case played by

$$\left.\begin{aligned}\vec{\psi}\left(\left|\vec{x}-\vec{x}'\right|\right) &= \int \varphi\left(\left|\vec{x}-\vec{x}''\right|\right)\varphi\left(\left|\vec{x}'-\vec{x}''\right|\right)\vec{x}''\,dv'' \\ &= \frac{e^{-\kappa\left|\vec{x}-\vec{x}'\right|}}{8\pi\kappa}\cdot\frac{\vec{x}+\vec{x}'}{2},\end{aligned}\right\}\text{(A 24)}$$

$$\left.\begin{aligned}\operatorname{rot}\vec{\psi}\left(\left|\vec{x}-\vec{x}'\right|\right) &= -\frac{1}{2}\varphi\left(\left|\vec{x}-\vec{x}'\right|\right)\cdot\left(\vec{x}-\vec{x}'\right)\wedge\frac{\vec{x}+\vec{x}'}{2} \\ &= -\frac{1}{2}\varphi\left(\left|\vec{x}-\vec{x}'\right|\right)\cdot\vec{x}\wedge\vec{x}'.\end{aligned}\right\}\text{(A 25)}$$

To begin with, we get from (A 18) and (A 24), (A 25),

$$\left.\begin{aligned}&\int \vec{x}\wedge\left(\vec{\boldsymbol{F}}^{\circ}\boldsymbol{\wedge}\boldsymbol{V}^{\circ}\right)dv = \\ &= -\int \boldsymbol{N}'\boldsymbol{\wedge}\boldsymbol{N}''\operatorname{rot}'\left\{\vec{x}\,\varphi\left(\left|\vec{x}-\vec{x}'\right|\right)\varphi\left(\left|\vec{x}-\vec{x}''\right|\right)\right\}dv\,dv'\,dv'' \\ &= -\int \boldsymbol{N}'\boldsymbol{\wedge}\boldsymbol{N}''\operatorname{rot}'\vec{\psi}\left(\left|\vec{x}'-\vec{x}''\right|\right)dv'\,dv'' \\ &= \frac{1}{2}\int \boldsymbol{N}'\boldsymbol{\wedge}\boldsymbol{N}''\varphi\left(\left|\vec{x}'-\vec{x}''\right|\right)\left(\vec{x}'\wedge\vec{x}''\right)dv'\,dv''.\end{aligned}\right\}\text{(A 26)}$$

Further, from (A 20), we get by the same procedure

$$\begin{aligned}&\int \vec{x}\wedge\left(\vec{\boldsymbol{G}}^{\circ}\underset{\boldsymbol{\wedge}}{\wedge}\vec{\boldsymbol{U}}^{\circ}\right)dv = \\ &= -\frac{1}{2}\kappa^{2}\int\left(\vec{\boldsymbol{S}}'\boldsymbol{\wedge}\vec{\boldsymbol{S}}''\right)\left(\vec{x}'\wedge\vec{x}''\right)\varphi\left(\left|\vec{x}'-\vec{x}''\right|\right)dv'\,dv'' \\ &\quad -\frac{1}{2}\int\left[\left(\vec{\boldsymbol{S}}''\operatorname{grad}''\right)\boldsymbol{\wedge}\left(\vec{\boldsymbol{S}}'\operatorname{grad}''\right)\right]\left(\vec{x}'\wedge\vec{x}''\right)\varphi\left(\left|\vec{x}'-\vec{x}''\right|\right)dv'\,dv'' \\ &\quad -\int\left(\vec{x}''\wedge\vec{\boldsymbol{S}}'\right)\boldsymbol{\wedge}\left(\vec{\boldsymbol{S}}''\operatorname{grad}''\varphi\right)dv'\,dv'' \\ &\quad -\int\left(\vec{x}\wedge\vec{\boldsymbol{S}}'\right)\boldsymbol{\wedge}\operatorname{div}\left(\operatorname{grad}'\varphi\boldsymbol{\wedge}\vec{\boldsymbol{U}}^{\circ}\right)dv\,dv';\end{aligned}$$

the second and third terms together give

$$\frac{1}{2}\int\left[\left(\vec{x}'-\vec{x}''\right)\wedge\vec{\boldsymbol{S}}'\right]\boldsymbol{\wedge}\left(\vec{\boldsymbol{S}}''\,\mathrm{grad}''\,\varphi\right)dv'\,dv''+\frac{1}{2}\int\left(\vec{x}'\wedge\vec{x}''\right)\left(\vec{\boldsymbol{S}}''\,\mathrm{grad}''\right)\boldsymbol{\wedge}\left(\vec{\boldsymbol{S}}'\,\mathrm{grad}'\right)\varphi\,dv'\,dv'',$$

while the last becomes

$$-\int\vec{\boldsymbol{S}}'\overset{\wedge}{\boldsymbol{\wedge}}\left(\mathrm{grad}'\,\varphi\wedge\vec{\boldsymbol{U}}^{\circ}\right)dv\,dv'=\int\vec{\boldsymbol{S}}'\overset{\wedge}{\boldsymbol{\wedge}}\left[\mathrm{grad}'\,\varphi\left(\left|\vec{x}-\vec{x}'\right|\right)\wedge\left(\vec{\boldsymbol{S}}''\wedge\mathrm{grad}''\,\varphi\left(\left|\vec{x}-\vec{x}''\right|\right)\right)\right]dv\,dv'\,dv''$$

$$=\int\left(\vec{\boldsymbol{S}}'\overset{\wedge}{\boldsymbol{\wedge}}\vec{\boldsymbol{S}}''\right)\left(\mathrm{grad}'\,\varphi\left(\left|\vec{x}-\vec{x}'\right|\right)\right)\cdot\mathrm{grad}''\,\varphi\left(\left|\vec{x}'-\vec{x}''\right|\right)dv\,dv'\,dv''$$

$$-\int\left[\vec{\boldsymbol{S}}'\wedge\mathrm{grad}''\,\varphi\left(\left|\vec{x}-\vec{x}''\right|\right)\right]\boldsymbol{\wedge}\left[\vec{\boldsymbol{S}}''\cdot\mathrm{grad}'\,\varphi\left(\left|\vec{x}-\vec{x}'\right|\right)\right]dv\,dv'\,dv''$$

$$=-\int\vec{\boldsymbol{S}}'\overset{\wedge}{\boldsymbol{\wedge}}\vec{\boldsymbol{S}}''\,\Delta'\,\psi\,dv'\,dv''-\int\left(\vec{\boldsymbol{S}}'\wedge\mathrm{grad}'\right)\boldsymbol{\wedge}\left(\vec{\boldsymbol{S}}''\,\mathrm{grad}''\right)\psi\,dv'\,dv'', \qquad \text{by (A 15)}$$

$$=-\int\vec{\boldsymbol{S}}'\overset{\wedge}{\boldsymbol{\wedge}}\vec{\boldsymbol{S}}''\left(\kappa^2\psi-\varphi\right)dv'\,dv''-\frac{1}{2}\int\left(\vec{\boldsymbol{S}}'\wedge\mathrm{grad}'\right)\boldsymbol{\wedge}\left[\vec{\boldsymbol{S}}''\cdot\left(\vec{x}'-\vec{x}''\right)\right]\varphi\,dv'\,dv'', \qquad \text{by (A 17) and (A 16)}$$

$$=-\int\vec{\boldsymbol{S}}'\overset{\wedge}{\boldsymbol{\wedge}}\vec{\boldsymbol{S}}''\left(\kappa^2\psi-\frac{\varphi}{2}\right)dv'\,dv''-\frac{1}{2}\int\left(\vec{\boldsymbol{S}}'\wedge\mathrm{grad}'\,\varphi\right)\boldsymbol{\wedge}\left[\vec{\boldsymbol{S}}''\cdot\left(\vec{x}'-\vec{x}''\right)\right]dv'\,dv''$$

$$=\int\left(\vec{\boldsymbol{S}}'\overset{\wedge}{\boldsymbol{\wedge}}\vec{\boldsymbol{S}}''\right)\left(\frac{\varphi}{2}-\kappa^2\psi\right)dv'\,dv''+\frac{1}{2}\int\left[\vec{\boldsymbol{S}}'\wedge\left(\vec{x}'-\vec{x}''\right)\right]\boldsymbol{\wedge}\left(\vec{\boldsymbol{S}}''\,\mathrm{grad}''\,\varphi\right)dv'\,dv'',$$

since $\mathrm{grad}'\,\varphi\left(\left|\vec{x}'-\vec{x}''\right|\right)$ is proportional to $\vec{x}'-\vec{x}''$; altogether, we thus get

$$\left.\begin{aligned}\int\vec{x}\wedge\left(\vec{\boldsymbol{G}}^{\circ}\boldsymbol{\wedge}\vec{\boldsymbol{U}}^{\circ}\right)dv=\int\left(\vec{\boldsymbol{S}}'\overset{\wedge}{\boldsymbol{\wedge}}\vec{\boldsymbol{S}}''\right)\left(\frac{\varphi}{2}-\kappa^2\psi\right)dv'\,dv''-\frac{\kappa^2}{2}\int\left(\vec{\boldsymbol{S}}'\boldsymbol{\wedge}\vec{\boldsymbol{S}}''\right)\left(\vec{x}'\wedge\vec{x}''\right)\varphi\,dv'\,dv''\\+\frac{1}{2}\int\left(\vec{x}'\wedge\vec{x}''\right)\left(\vec{\boldsymbol{S}}''\,\mathrm{grad}''\right)\boldsymbol{\wedge}\left(\vec{\boldsymbol{S}}'\,\mathrm{grad}'\right)\varphi\,dv'\,dv''.\end{aligned}\right\}\quad\text{(A 27)}$$

From (A 26), (A 27), and taking account of (A 15), we obtain for the contribution of the vector meson field alone to $\vec{M}_{(0)}$

$$\left.\begin{aligned}\vec{M}^{\text{vect}}_{(0)} &= \frac{e}{4\hbar c}\int\left\{\vec{\boldsymbol{S}}'\,\overset{\wedge}{\boldsymbol{\wedge}}\,\vec{\boldsymbol{S}}''\right\}_{\mathbf{3}}\left(1-\kappa\left|\vec{x}'-\vec{x}''\right|\right)\varphi\,dv'\,dv''\\ &-\frac{e}{4\hbar c}\int\left\{\boldsymbol{N}'\boldsymbol{\wedge}\boldsymbol{N}''+\kappa^2\,\vec{\boldsymbol{S}}'\boldsymbol{\wedge}\vec{\boldsymbol{S}}''\right\}_{\mathbf{3}}\left(\vec{x}'\wedge\vec{x}''\right)\varphi\,dv'\,dv''\\ &+\frac{e}{4\hbar c}\int\left(\vec{x}'\wedge\vec{x}''\right)\left\{\left(\vec{\boldsymbol{S}}''\,\mathrm{grad}''\right)\boldsymbol{\wedge}\left(\vec{\boldsymbol{S}}'\,\mathrm{grad}'\right)\right\}_{\mathbf{3}}\varphi\,dv'\,dv'';\end{aligned}\right\}\quad\text{(A 28)}$$

owing to the factors $\vec{x}'\wedge\vec{x}''$, the last two terms are of the exchange type only, while the first also involves a proper magnetic moment given by (62).

Finally, we derive from (A 22), for the pseudoscalar field,

$$\int\vec{x}\wedge\left(\vec{\boldsymbol{\Gamma}}^{\circ}\boldsymbol{\wedge}\boldsymbol{\Psi}^{\circ}\right)dv$$

$$\begin{aligned}&=\int\left(\vec{x}''\wedge\vec{\boldsymbol{P}}''\right)\boldsymbol{\wedge}\left(\vec{\boldsymbol{P}}'\,\mathrm{grad}'\,\varphi\right)dv'\,dv''\\ &\quad-\frac{1}{2}\int\left(\vec{\boldsymbol{P}}''\,\mathrm{grad}''\right)\boldsymbol{\wedge}\left(\vec{\boldsymbol{P}}'\,\mathrm{grad}'\right)\left(\vec{x}'\wedge\vec{x}''\right)\varphi\,dv'\,dv''\\ &=-\int\left[\left(\vec{x}'-\vec{x}''\right)\wedge\vec{\boldsymbol{P}}''\right]\boldsymbol{\wedge}\left(\vec{\boldsymbol{P}}'\,\mathrm{grad}'\,\varphi\right)dv'\,dv''+\frac{1}{2}\int\vec{\boldsymbol{P}}''\,\overset{\wedge}{\boldsymbol{\wedge}}\,\vec{\boldsymbol{P}}'\,\varphi\,dv'\,dv''\\ &\quad-\frac{1}{2}\int\left(\vec{x}'\wedge\vec{x}''\right)\left(\vec{\boldsymbol{P}}''\,\mathrm{grad}''\right)\boldsymbol{\wedge}\left(\vec{\boldsymbol{P}}'\,\mathrm{grad}'\right)\varphi\,dv'\,dv''.\end{aligned}$$

With the notation $\vec{x}_0\equiv\dfrac{\vec{x}'-\vec{x}''}{\left|\vec{x}'-\vec{x}''\right|}$, the first term may be written

$$\frac{1}{2}\int\left[\left(\vec{x}_0\wedge\vec{\boldsymbol{P}}''\right)\boldsymbol{\wedge}\left(\vec{\boldsymbol{P}}'\,\vec{x}_0\right)+\left(\vec{x}_0\wedge\vec{\boldsymbol{P}}'\right)\boldsymbol{\wedge}\left(\vec{\boldsymbol{P}}''\,\vec{x}_0\right)\right]\left(1+\kappa\left|\vec{x}'-\vec{x}''\right|\right)\varphi\,dv'\,dv''$$

$$=\frac{1}{2}\int\left[\left(\vec{\boldsymbol{P}}'\,\vec{x}_0\right)\boldsymbol{\wedge}\vec{\boldsymbol{P}}''-\vec{\boldsymbol{P}}'\boldsymbol{\wedge}\left(\vec{\boldsymbol{P}}''\,\vec{x}_0\right)\right]\wedge\vec{x}_0\left(1+\kappa\left|\vec{x}'-\vec{x}''\right|\right)\varphi\,dv'\,dv'';$$

with the help of the vector relation $\left(\vec{a}\wedge\vec{b}\right)\wedge\vec{c}=\left(\vec{a}\,\vec{c}\right)\vec{b}-\vec{a}\left(\vec{b}\,\vec{c}\right)$, which holds when $\vec{b}$ and $\vec{c}$ are commutable, this transforms into

$$\frac{1}{2}\int\left[\left(\vec{\boldsymbol{P}}'\,\overset{\wedge}{\boldsymbol{\wedge}}\,\vec{\boldsymbol{P}}''\right)\wedge\vec{x}_0\right]\wedge\vec{x}_0\left(1+\kappa\left|\vec{x}'-\vec{x}''\right|\right)\varphi\,dv'\,dv'',$$

or, using again the same relation,

$$-\frac{1}{2}\int\left\{\vec{P}'\wedge\vec{P}''-\left[\left(\vec{P}'\wedge\vec{P}''\right)\vec{x}_0\right]\vec{x}_0\right\}\left(1+\kappa\left|\vec{x}'-\vec{x}''\right|\right)\varphi\, dv'\, dv''.$$

Hence

$$\left.\begin{aligned}\vec{M}_{(0)}^{\text{ps-scal}} = &-\frac{e}{4\hbar c}\int\Big\{\left\{\vec{P}'\wedge\vec{P}''\right\}_3\kappa\left|\vec{x}'-\vec{x}''\right|\\ &-\left[\left\{\vec{P}'\wedge\vec{P}''\right\}_3\vec{x}_0\right]\vec{x}_0\left(1+\kappa\left|\vec{x}'-\vec{x}''\right|\right)\Big\}\varphi\, dv'\, dv''\\ &-\frac{e}{4\hbar c}\int\left(\vec{x}'\wedge\vec{x}''\right)\left\{\left(\vec{P}''\,\text{grad}''\right)\wedge\left(\vec{P}'\,\text{grad}'\right)\right\}_3\varphi\, dv'\, dv'';\end{aligned}\right\}\quad (\text{A } 29)$$

here also the first integral only contributes to the proper magnetic moment, *viz.* the moment (63).

It now again appears that, to the first order in the nuclear velocities, the last term in (A 29) just compensates the corresponding term of (A 28), so that, using NF (3, 4, 24) and the notation (59), and taking into account that $f_2^2 = g_2^2$, we find for the exchange moment in the mixed theory the expression (60). It will be noticed that the exchange moment is not translation invariant; if the origin is displaced by $\vec{a}$, the exchange moment changes by $\frac{1}{2}\vec{a}\wedge\vec{J}$.

## Expression of the proper magnetic moments of nucleons for special source distributions.

We shall here briefly indicate how the formulae (66), (67), (68) for the "Landé factor" $\mu$ of the proper magnetic moment may be evaluated under simple assumptions over the source density function $D(r)$. The integrals occurring in these formulae are of the form

$$\mathcal{F} = \int D(r')\, dv' \int D(r'')\, dv''\, \frac{F(\kappa\varrho)}{4\pi\varrho},$$

with

$$\varrho = \sqrt{r'^2 + r''^2 - 2r'r''u},$$

$u$ denoting the cosine of the angle between $\vec{x}'$ and $\vec{x}''$. Introducing suitable polar coordinates, we obtain

$$\mathcal{F} = \int\frac{D(r')\, dv'}{2\kappa r'}\int D(r'')\, r''\, dr''\int_{\kappa|r'-r''|}^{\kappa(r'+r'')} F(y)\, dy. \qquad (\text{A } 30)$$

For the contributions from the vector and pseudoscalar meson field, respectively, we have

$$F^{\text{vect}}(y) = (1-y)\,e^{-y}, \quad F^{\text{ps-scal}}(y) = (1-2y)\,e^{-y}. \quad \text{(A 31)}$$

Consider a uniform source distribution on the surface of a sphere of radius $r_0$; the corresponding density function may be expressed by

$$D(r) = \frac{\delta(r-r_0)}{4\pi r_0^2} \quad \text{(A 32)}$$

and (A 30) then reduces to

$$\mathscr{F}^{\text{sup}} = \frac{1}{8\pi r_0^2 \kappa}\int_0^{2\kappa r_0} F(y)\,dy. \quad \text{(A 33)}$$

In this way, putting $\varepsilon \equiv \kappa r_0$, we obtain

$$\left.\begin{aligned}
\mu^{\text{vect}} &= \frac{g_2^2}{4\pi\hbar c}\cdot\frac{M}{M_m}\cdot\frac{2}{\varepsilon}\,e^{-2\varepsilon} \\
\mu^{\text{ps-scal}} &= \frac{f_2^2}{4\pi\hbar c}\,\frac{M}{M_m}\cdot\frac{1}{3\varepsilon^2}\left[e^{-2\varepsilon}(1+4\varepsilon)-1\right] \\
\mu &= \frac{g_2^2}{4\pi\hbar c}\,\frac{M}{M_m}\cdot\frac{1}{3\varepsilon^2}\left[e^{-2\varepsilon}(1+10\varepsilon)-1\right].
\end{aligned}\right\} \quad \text{(A 34)}$$

For small values of $\varepsilon$, the functions of $\varepsilon$ occurring in (A 34) approximately become

$$\frac{2}{\varepsilon}-4, \qquad \frac{2}{3\varepsilon}-2, \qquad \frac{8}{3\varepsilon}-6,$$

respectively.

In the case of a uniform volume distribution of the sources within a sphere of radius $r_0$, we get from (A 30), (A 31), by a straightforward calculation,

$$\left.\begin{aligned}
\mu^{\text{vect}} &= \frac{g_2^2}{4\pi\hbar c}\,\frac{M}{M_m}\cdot\frac{18}{\varepsilon^6}\left[-\frac{\varepsilon^3}{3}+\varepsilon^2-2+e^{-2\varepsilon}(\varepsilon+1)(\varepsilon^2+2\varepsilon+2)\right] \\
\mu^{\text{ps-scal}} &= \frac{f_2^2}{4\pi\hbar c}\,\frac{M}{M_m}\cdot\frac{3}{\varepsilon^6}\left[-2\varepsilon^3+5\varepsilon^2-9+e^{-2\varepsilon}(\varepsilon+1)(4\varepsilon^2+9\varepsilon+9)\right] \\
\mu &= \frac{g_2^2}{4\pi\hbar c}\,\frac{M}{M_m}\cdot\frac{3}{\varepsilon^6}\left[-4\varepsilon^3+11\varepsilon^2-21+e^{-2\varepsilon}(\varepsilon+1)(10\varepsilon^2+21\varepsilon+21)\right]
\end{aligned}\right\} \quad \text{(A 35)}$$

For small values of $\varepsilon$, the approximate values of the functions of $\varepsilon$ occurring in (A 35) are

$$\frac{36}{5\varepsilon}, \qquad \frac{22}{5\varepsilon}, \qquad \frac{58}{5\varepsilon},$$

respectively.

The formulae (A 34), (A 35) have been obtained by SERPE in another way; for a numerical discussion, the reader may be referred to SERPE's paper.

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## Correction:

p. 6, 29th line, 'it is out of question' read 'it is excluded'.

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